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Time-Dependent Approach to Nonrelativistic Multichannel Scattering (*).

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Summary. — The mathematical problems occurring in nonrelativistic multichannel scattering theory are investigated with the help of Kato's work on the time-dependent abstract scattering theory. We concentrate mainly on two-body reactions in multiparticle systems; a physically intuitive form is rigorously derived for the corresponding wave-operators and a partial asymptotic completeness is proved below the three body threshold. We also investigate the relevance of the time-dependent framework for a complete detailed study of three-particle systems. This approach also leads us to establish some auxiliary results on multiparticle bound states.

Introduction.

The nonrelativistic scattering theory for multiparticle systems as formulated by EKSTEIN (1) and JAUCH (2) has raised two important mathematical problems:

a) Existence and computation of scattering states.

b) Completeness of these scattering states.

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(1) H. EKSTEIN: *Phys. Rev.*, **101**, 2, 880 (1956).

(2) J. M. JAUCH: *Helv. Phys. Acta*, **31**, 661 (1958).

The usual two-body techniques such as the Lippman-Schwinger equation or the representation theorem ([3]) for wave operators as strongly convergent integrals are not powerful enough to solve completely these problems; they can, at most, be successful in providing existence proofs for *a*). The general reasons for this are, first, that two-body potentials are not sufficiently « small » compared to the total kinetic energy; and, second, that the existence of rearrangement collisions gives a nondiagonal form to the *S*-matrix. Existing work on two-body scattering problems (*e.g.* ([4,5])) suggests that *a*) and *b*) both require integral equations with compact kernels in a suitable Banach space. Such equations have been proposed by FADDEEV ([6]) and YAKUBOVSKY ([7]). They have been used by FADDEEV himself ([6]) and HEPP ([8]) to show that *a*) and *b*) can be solved for the N-particle problem when the p-particle scattering amplitudes, $p < N$, are known and sufficiently smooth. However for low-energy problems and large N these equations seem to be more complicated than needed; a simpler formulation can be expected from the fact that in atomic or nuclear problems many particles remain bound together; consequently only few-body reactions must be investigated. One of our aims is to illustrate the possibility of such a formulation in the simplest case of two-body reactions. We show (Sect. 3), using a time-dependent version of the distorted-wave theory, that the whole scattering process consists of two steps: first an elastic scattering, due to the long-range effect of the potential, between the two composite particles; and second a short-range inelastic process. For this we use a decomposition of the potentials into parts « within » and « between » the various channels; each of these parts has sufficiently nice mathematical properties so that existence of wave operators and a partial asymptotic completeness below the three-body threshold can be proved for a rather large class of local as well as nonlocal interactions (Sect. 4).

In order to improve our understanding of the mathematical mechanisms governing multichannel scattering, and as a first indication of a possible time-dependent treatment of three-body reactions along the lines described above, we shall also study the whole *S*-matrix of a three-particle system (Sect. 5). A new type of (nonsingular) perturbation must then be introduced; however

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([3]) J. M. JAUCH and I. ZINNES: *Nuovo Cimento*, **11**, 553 (1959).

([4]) T. IKEBE: *Arch. Rat. Mech. Anal.*, **5**, 1 (1960).

([5]) L. D. FADDEEV: *Trud. Math. Inst. Steklov*, **73**, 292 (1964).

([6]) L. D. FADDEEV: *Mathematical Aspects of the Three-Body Problem in the Quantum Scattering Theory. Israël Programme for Scientific Translation, Jerusalem, 1965* (London).

([7]) O. A. YAKUBOVSKY: *Sov. Journ. Nucl. Phys.*, **5**, 937 (1967).

([8]) K. HEPP: *On the quantum-mechanical N-body problem* (to appear in *Helv. Phys. Acta*).

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an unexpected phenomenon can break unitarity in situations about which we cannot yet say whether they are pathological.

The first Section contains the kinematics. In Sect. 2 we state some preliminary qualitative results on N-particle Hamiltonians and bound-state wave functions.

1. – Kinematics and dynamics.

We consider an N-particle system. We shall denote by $D = \{C_1; C_2; \ldots; C_k\}$ a decomposition of this system into k disjoint clusters (or fragments), writing $D = C$ if C is the only p-particle fragment with $p > 1$ and $D = \{N\}$ for the trivial decomposition into one cluster which we also denote by $\{N\}$. We shall write p_C and x_C for the momentum and position vector of the centre of mass of cluster C, and m_C for its total mass; the internal momentum and position vectors for the particles of cluster C subjected to the center-of-mass condition $x_C = p_C = 0$ will be designed respectively by p^C and x^C.

The total interaction between particles of cluster C will be denoted by V^C. We shall assume that there is no external field acting on $\{N\}$; in particular $V^i = 0$ for each particle i. The total Hamiltonian for a subsystem $C \subseteq \{N\}$ is then

$$H'^C = \sum_{i \in C} \frac{|p_i|^2}{2m_i} + V^C = \frac{|p_C|^2}{2m_C} + H_0^C + V^C$$

and in the center-of-mass system of C

$$H^C = H_0^C + V^C .$$

An easy calculation then shows

$$H_0^C = H_0^{C_1} + H_0^{C_2} + \frac{|p^{(C_1,C_2)}|^2}{2m^{(C_1,C_2)}} , \tag{1}$$

where C_1 and C_2 are arbitrary disjoint clusters whose union is C, their reduced mass being $m^{(C_1,C_2)}$ and their relative momentum $p^{(C_1,C_2)}$ being defined by

$$p^{(C_1,C_2)} = \frac{m_{C_1} p_{C_2} - m_{C_2} p_{C_1}}{m_{C_1} + m_{C_2}} ;$$

we shall use the notation

$$H_0^{(C_1,C_2)} = \frac{|p^{(C_1,C_2)}|^2}{2m^{(C_1,C_2)}} .$$

The conjugate co-ordinates of $p_j^{(C_1,C_2)}$, $j=1, 2, 3$, are $x_j^{(C_1,C_2)}$ where $x^{(C_1,C_2)} = x_{C_1} - x_{C_2}$. By writing $C_1 = C_3 \cup C_4$ etc., we get a (nonunique) set (p_j^C), $j=1, 2, \ldots, 3(p-1)$, of momentum co-ordinates for p^C and the corresponding set (x_i^C) of conjugate position co-ordinates for x^C.

Such sets of canonical variables satisfy the canonical commutation relations

$$[x_i^C, x_j^C] = [p_i^C, p_j^C] = 0 , \tag{2}$$

$$[x_i^C, p_j^C] = + i\delta_{ij} . \tag{3}$$

The representation space of the system C is the Hilbert space $\mathscr{H}'^C = L^2(x_C, x^C)$ which can be written as [9]

$$\mathscr{H}'^C = \mathscr{H}_C \otimes \mathscr{H}^C , \tag{4}$$

with $\mathscr{H}_C = L^2(x_C)$ and $\mathscr{H}^C = L^2(x^C)$. Furthermore if C_1 and C_2 are disjoint clusters whose union is C one has

$$\mathscr{H}^C = \mathscr{H}^{C_1} \otimes \mathscr{H}^{C_2} \otimes \mathscr{H}^{(C_1,C_2)} , \tag{5}$$

where $\mathscr{H}^{(C_1,C_2)} = L^2(x^{(C_1,C_2)})$. Furthermore owing to the kinematical independence of C_1 and C_2 expressed by the commutation relations

$$[x_i^{C_1}, p_j^{C_2}] = [x_i^{C_1}, p_j^{(C_1,C_2)}] = 0 ,$$

multiplication operators on $\mathscr{H}^C$ like $x_i^{C_1}$ or derivation operators like $p_j^{C_1} = -i(\partial/\partial x_j^{C_1})$ are obtained by ampliation of the same multiplication or derivation operator on $\mathscr{H}^{C_1}$; since they generate the algebra of bounded operators on $\mathscr{H}^{C_1}$ we shall use the convention to denote by the same symbol the operators A^{C_1} on $\mathscr{H}^{C_1}$ and corresponding ampliated operators $A^{C_1} \otimes I_{\mathscr{H}^{C_2}} \otimes I_{\mathscr{H}^{(C_1,C_2)}}$ on $\mathscr{H}^C$.

A channel of the system is a set

$$\alpha = \{b(C_1), b(C_2), \ldots, b(C_k)\}$$

where $C_1, C_2, \ldots, C_k$ are disjoint clusters such that $\bigcup_{i=1}^{k} C_i = \{N\}$ and $b(C)$ denotes an eigensubspace of $H^{\{C\}}$ corresponding to a bound state of the system C with binding energy $-E^{b(C)}$; if C contains only one particle then $b(C)$ specifies this particle and $E^{b(C)} = 0$. Without restricting the generality we shall always assume that $b(C)$ is one-dimensional eigensubspace of H^C. Let $P^{b(C)}$ be the projection operator in $\mathscr{H}^C$ on the wave function of the bound state $b(C)$; owing to our convention $P^{b(C)}$ also denotes the corresponding projector in

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[9] J. M. COMBES: *Properties of some connected kernels in multiparticle systems* (preprint).

$\mathscr{H}^{\{N\}}$; then the projection operator in channel α is

$$P^{\alpha} = \prod_{C \in D(\alpha)} P^{b(C_i)} \tag{6}$$

($P^{b(C)} = 1$ if C contains only one particle) and the projection operator upon its orthogonal complement is

$$Q^{\alpha} = I - P^{\alpha} ; \tag{7}$$

here $D(\alpha) = \{C_1; C_2; \ldots; C_k\}$ is the cluster decomposition associated to α. Let $V^D = \sum_{C \in D} V^C$; then the channel interaction V^{α} is defined as

$$V^{\alpha} = V^{\{N\}} - V^{D(\alpha)} . \tag{8}$$

The channel energy is $E^{\alpha} = \sum_{C \in D(\alpha)} E^{b(C)}$ and the free Hamiltonian in channel α is given by

$$H_0^{\alpha} = H^{D(\alpha)} P^{\alpha} ,$$

where $H^D = H_0^{\{N\}} + V^D$; let us remark that $[H^{D(\alpha)}, P^{\alpha}] = 0$. For two-body channels $D(\alpha) = \{C_1, C_2\}$, one has then owing to (1)

$$H_0^{\alpha} = \left[\frac{|p^{(C_1, C_2)}|^2}{2m^{(C_1, C_2)}} + E^{\alpha} \right] P^{\alpha} .$$

The Hamiltonian H_0^{α} is a self-adjoint operators whose spectrum lies on the axis $[E^{\alpha}, +\infty[$ with $E^{\alpha} = \sum_{b(C) \in \alpha} E^{b(C)}$. We define the p-body threshold Λ_C^p of a Hamiltonian $H^{\{N\}}$ as $\inf E^{\alpha}$ when $D(\alpha)$ runs through the set of p fragment cluster decompositions of the subsystem C. Under the condition that for any subsystem C there exists bound states of C below the continuum $[\Lambda_C, +\infty[$ (see theorem 1 below) one has

$$\Lambda_C^2 < \Lambda_C^3 < \ldots \Lambda_C^p < \Lambda_C^{p+1} < \ldots .$$

The wave operators for channel α are defined by

$$\Omega_{\pm}^{\alpha} = \lim_{t \to \pm\infty} \exp [iH^{\{N\}} t] \exp [-iH_0^{\alpha} t] P^{\alpha} . \tag{9}$$

Disregarding existence problems, some of which will be treated below, we state the elementary properties of $\Omega_{\pm}^{\alpha}$:

a) They are partially isometric operators from $P^{\alpha} \mathscr{H}^{\{N\}}$ into $\mathscr{H}^{\{N\}}$; more precisely we have $\Omega_{\pm}^{\alpha} \mathscr{M}_a(H_0^{\alpha}) \subset \mathscr{M}_a(H^{\{N\}})$ (here $\mathscr{M}_a(H_0^{\alpha})$ is obviously equal to $P^{\alpha} \mathscr{H}^{\{N\}}$; see Appendix I for the definition of $\mathscr{M}_a$).

b) They satisfy the intertwining property

$$\Omega_{\pm}^{\alpha} H_0^{\alpha} = H^{\{N\}} \Omega_{\pm}^{\alpha}$$

and a similar equality where H_0^{α} and $H^{\{N\}}$ are replaced by elements of their respective spectral families.

c) For any α and β one has $(\Omega_{\pm}^{\alpha})^* \Omega_{\pm}^{\beta} = \delta_{\alpha\beta} P^{\alpha}$.

This property known as « asymptotic orthogonality of channels » is proved in ([2]) for general multichannel systems. For two-body channels we derive it by a different method showing a slightly stronger result. Let α and β be different two-body channels. If $D(\alpha) \neq D(\beta)$ then $P^{\alpha} P^{\beta}$ is a compact operator on $\mathscr{H}^{\{N\}}$ owing to theorem 3 below. Consequently by a well-known property of compact operators

$$(\exp[-iH_0^{\alpha}t] P^{\alpha} f, \exp[-iH_0^{\beta}t] P^{\beta} g) = (\exp[-iH_0^{\alpha}t] f,\ P^{\alpha} P^{\beta} \exp[-iH_0^{\beta}t] g)$$

tends to zero uniformly in f or g ($\|f\|, \|g\| \leqslant 1$) as $t \to \pm\infty$. If $D(\alpha) = D(\beta)$ one has $P^{\alpha} P^{\beta} = 0$ and then *c*) is obvious. As a result for any self-adjoint operator A such that $\Omega_{\pm}(A; H_0^{\alpha})$ and $\Omega_{\pm}(A; H_0^{\beta})$ exist, the ranges of those wave operators are orthogonal.

The S-matrix is then the set of operators

$$S_{\alpha\beta} = (\Omega_{+}^{\beta})^* \Omega_{-}^{\alpha}.$$

Let us now explain what we mean by unitarity of the S-matrix below some energy Λ and in particular below some threshold $\Lambda_{\{N\}}^{v}$ of $H^{\{N\}}$. If A is some self-adjoint operator and $E(\Lambda)$ its spectral family we define

$$\mathscr{M}_a^{\Lambda}(A) = \mathscr{M}_a(E([-\infty, \Lambda[)\, A)\,,$$

and the invariant subspace of $\mathscr{M}_a^{\Lambda}(H^{\{N\}})$

$$M_{\pm}^{\Lambda} = \bigoplus_{\alpha} \Omega_{\pm}^{\alpha} \mathscr{M}_a^{\Lambda}(H_0^{\alpha})\,.$$

M_{+}^{Λ} (respectively M_{-}^{Λ}) consists of all the vectors in $\mathscr{M}_a^{\Lambda}(H^{\{N\}})$ which behave asymptotically when $t \to +\infty$ (respectively $-\infty$) like

$$\sum_{\alpha} \exp[-iH_0^{\alpha}t] f_{+}^{\alpha} \text{ (respectively } f_{-}^{\alpha}) \quad f_{\pm}^{\alpha} \in \mathscr{M}_a^{\Lambda}(H_0^{\alpha})\,,$$

this decomposition being unique owing to property *c*). If $\Lambda \leqslant \Lambda_{\{N\}}$ the $f_{\pm}^{\alpha}$ describe a set of $q < p$ bodies with total kinetic energy (in the center-of-mass system) smaller than $\Lambda - E^{\alpha}$. If $f \in M_{\pm}^{\Lambda}$ one has

$$f_{\pm}^{\alpha} = (\Omega_{\pm}^{\alpha})^* f\,, \qquad f = \sum_{\alpha} \Omega_{\pm}^{\alpha} f_{\pm}^{\alpha} \tag{10}$$

and if $f \in M_{+}^{\Lambda} \cap M_{-}^{\Lambda}$ asymptotic orthogonality of channels gives

$$\|f\|^2 = \sum_{\alpha} \|f_{-}^{\alpha}\|^2 = \sum_{\alpha} \|f_{+}^{\alpha}\|^2 \qquad f_{\pm}^{\alpha} \in \mathscr{M}_a^{\Lambda}(H_0^{\alpha})\,.$$

Since $f_{+}^{\beta} = \sum_{\alpha} S_{\alpha\beta} f_{-}^{\alpha}$ the above equality expresses the unitarity of the S-matrix below Λ. In order that it hold for any set of initial data $(f_{-}^{\alpha} \in \mathscr{M}_a^{\Lambda}(H_0^{\alpha}))$ it is then sufficient that $M_{-}^{\Lambda} \subset M_{+}^{\Lambda}$. This conditions also is necessary since the contrary would imply $\sum_{\beta} \|f_{+}^{\beta}\|^2 < \sum_{\alpha} \|f_{-}^{\alpha}\|^2$ which expresses a loss of information on the final data. This, together with time-reversal invariance, implies that the S-matrix is unitary below Λ if and only if $M_{+}^{\Lambda} = M_{-}^{\Lambda}$; this condition holds in particular if the wave operator $\Omega_{\pm}^{\alpha}$ satisfy the asymptotic completeness relation below Λ

$$M_{+}^{\Lambda} = M_{-}^{\Lambda} = \mathscr{M}_a^{\Lambda}(H^{\{N\}})\,. \tag{11}$$

In Sect. **4** we shall prove (11) for $\Lambda = \Lambda_{\{N\}}^{3}$.

2. – Potentials and bound states.

We now establish some elementary properties of the multiparticle Hamiltonians H^{σ} which are essential for later purposes. In particular we shall need to know their domain, their essential spectrum and the asymptotic decrease in configuration space of their bound-state wave functions. We begin with a classical result of KATO ([10a]):

Theorem K.0. Let H_0 be a self-adjoint operator on the Hilbert space $\mathscr{H}$ and V a symmetric H_0-bounded operator on $\mathscr{H}$ (this means that $\mathscr{D}(H_0) \subset \mathscr{D}(V)$ and that for all f in $\mathscr{D}(H_0)$ the inequality

$$\|Vf\| \leqslant a\|H_0 f\| + b\|f\| \tag{12}$$

([10]) T. KATO: *Perturbation Theory for Linear Operators*, *a*) Ch. IV, Sect. **1**; *b*) Ch. X, Sect. **4**; *c*) Ch. X, Sect. **3** (Berlin, Heidelberg, New York, 1966).

holds for some positive constants a and b). Then if a can be chosen smaller than one, the operator H_0+V is self-adjoint and its domain coincides with $\mathscr{D}(H_0)$.

The greatest lower bound of all possible constants a such that (12) is satisfied is called the H_0-bound of V. If a can be chosen as small as we want, we shall say that V is $(H_0\text{-}\varepsilon)$-bounded. We remember that $\mathscr{D}(H_0)$ is a Hilbert space for the scalar product

$$(f, g)_{\mathscr{D}(H_0)} = (H_0 f, H_0 g)_{\mathscr{H}} + (f, g)_{\mathscr{H}} .$$

A special class of $(H_0\text{-}\varepsilon)$-bounded operators consists of compact operators from $\mathscr{D}(H_0)$ to $\mathscr{H}$; they are said to be H_0-compact in $\mathscr{H}$. Some properties of H_0^c-compact potentials are studied in ([11]) where the following theorem is proved:

Theorem 1. Let $V^c = \sum_{c' \subset c} I^{c'}$ where each $I^{c'}$ is symmetric and $H_0^{c'}$-compact in $\mathscr{H}^{c'}$. Then $H^c = H_0^c + V^c$ is self-adjoint and its domain coincides with $\mathscr{D}(H_0^c)$; its spectrum consists of a continuum $[\Lambda^c, +\infty[$, where Λ_c is the lowest many-body threshold of H^c, and in the complement of this continuum of eigenvalues with finite multiplicities which can accumulate at most at Λ_c.

Relative compactness is a very weak assumption on the potentials which is satisfied by a large class of local and velocity-dependent many-particle interactions. The fact that the eigenvalues of H^c below Λ_c are isolated and have finite multiplicities has important implications on the corresponding bound-state wave functions as shown by the next theorem.

Let $(n) = n_1, \ldots, n_{3(p-1)}$, be a set of positive integers and $|(n)| = \sum_i n_i$; we define $(x^c)^{\pm(n)} = (x_1^c)^{\pm n_1} \cdot (x_2^c)^{\pm n_2} \ldots (x_{3(p-1)}^c)^{\pm n_{3(p-1)}}$.

Theorem 2. Let V^c be a symmetric interaction with an H_0^c-bound smaller than one. Let n_0 be some positive integer; then if $[V^c, (x^c)^{(n)}](x^c)^{-(m)}$ is H_0^c-bounded for each (n) with $|(n)| \leqslant n_0$ and some (m) with $m_i \leqslant n_i \; \forall \, i$ and $|(m)| < |(n)|$, the wave functions of isolated eigenvalues of H^c having finite multiplicities are in $\mathscr{D}_{n_0}^c = \bigcap_{|(n)| \leqslant n_0} \mathscr{D}\big((x^c)^{(n)}(1 + H_0^c)\big)$.

Proof. We first show the theorem for $n_0 = 1$. Then $\mathscr{D}_1^c$ is a Banach space for the norm $\|f\|_1^c = \sup_{|(n)| \leqslant 1} \|(x^c)^{(n)}(1 + H_0^c) f\|_{\mathscr{H}^c}$.

Let $P^{b(c)}$ be the spectral projection operator of H^c associated to one of its isolated eigenvalue $E^{b(c)}$. Since $\Delta_1^c = \bigcap_{|(n)| \leqslant 1} \mathscr{D}\big((x^c)^{(n)}\big)$ is dense in $\mathscr{H}^c$ and $P^{b(c)}$ is a finite-rank operator, $P^{b(c)}\mathscr{H}^c$ is spanned by the vectors in $P^{b(c)}\Delta_1^c$; so that

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([11]) J. M. COMBES: *Comm. Math. Phys.*, **12**, 283 (1969).

in order to prove the theorem for $n_0 = 1$ it is sufficient to show that these vectors are in $\mathscr{D}_1^c$. The projection operator $P^{b(c)}$ can be written as

$$P^{b(c)} = \frac{1}{2i\pi} \int_\gamma G_c(z)\,\mathrm{d}z\,,$$

where $G_c(z) = (z - H^c)^{-1}$ and γ is some circle, contained in the resolvent set of H^c, enclosing $E^{b(c)}$ and no other point in the spectrum of H^c. Hence for $z \in \gamma$ the range of $G_c(z)$ is $\mathscr{D}(H^c)$. For any $f \in \Delta_1^c$ one has then

$$\|P^{b(c)} f\|_1^c \leqslant \sup_{|(n)|<1} \frac{1}{2\pi} \int_\gamma \|(x^c)^{(n)}(1 + z - V^c) G_c(z) f\|_{\mathscr{H}^c}\,\mathrm{d}z\,.$$

From our hypothesis on V^c we only need to show that

$$\|A^c (x^c)^{(n)} G_c(z) f\|_{\mathscr{H}^{\{c\}}} < \infty$$

for all (n) with $|(n)| \leqslant n_0$, all $z \in \gamma$ and any H_0^c-bounded operator A^c. This can be done easily if we are allowed to use the formal commutation rules:

$$[(x^c)^{(n)}, G_c(z)] = G_c(z)\big([(x^c)^{(n)}, H_0^c] + [(x^c)^{(n)}, V^c]\big) G_c(z)\,, \tag{13}$$

$$[(x^c)^{(n)}, H_0^c] = \sum_{\{j|n_j \geqslant 1\}} \frac{i(n_j - 1)}{m}\, p_j^c (x^c)^{(n)-(1_j)} + \sum_{\{j|n_j \geqslant 2\}} \frac{i(n_j - 1)(n_j - 2)}{m}\, (x^c)^{(n)-(2_j)}, \tag{14}$$

where $(k_j) = 0, \ldots, k_j, \ldots, 0$ and $(n) - (k_j) = n_1, \ldots, n_j - k_j, \ldots, n_{3(p-1)}$. Then owing to the fact that $[(x^c)^{(n)}, V^c]$ and p_j^c are H_0^c (and by theorem K.0 also H^c) bounded, we get

$$\sup_{|(n)| \leqslant 1} \|(x^c)^{(n)} G_c(z) f\|_{\mathscr{D}(H^c)} \leqslant B(f, f)_{\Delta_1^c}$$

and consequently for any H_0^c bounded operator A^c

$$\sup_{|(n)| \leqslant 1} \|A^c (x^c)^{(n)} G_c(z) f\|_{\mathscr{H}^c} \leqslant B'(f, f)_{\Delta_1^c}\,,$$

where $(g, f)_{\Delta_1^c} = \Big(\big(1 + \sum_i |x_i^c|\big) g,\ \big(1 + \sum_i |x_i^c|\big) f\Big)_{\mathscr{H}^c}$ makes Δ_1^c a Hilbert space.

In fact the commutation relations (13) and (14) can be rigorously established on the set $(H^c - z)\mathscr{S}^c$ where $\mathscr{S}^c$ is the space of infinitely differentiable functions with fast decrease; the inclusion $(H^c - z)\mathscr{S}^c \subset \Delta_1^c$ is easily established under our hypotheses and in order to validate the above inequalities it is

sufficient to show that $(H^c - z)\mathscr{S}^c$ is dense in Δ_1^c. Suppose then that there exists an $f \in \Delta_1^c$ such that $(f, (H^c - z)s)_{\Delta_1^c} = 0$ for all $s \in \mathscr{S}$ and let $u = G_c(z) \cdot \left(1 + \sum_i |x_i^c|\right) f$. Since $\mathscr{D}(H^c) = \mathscr{D}(H_0^c)$, the set $\mathscr{S}^c$ is dense in $\mathscr{D}(H^c)$ and consequently there exists $s_n \in \mathscr{S}^c$ such that $\|(H^c - z)(s_n - u)\|_{\mathscr{H}^c}$ tends to zero; but then $\left((H^c - z)s_n, \left(1 + \sum_i |x_i^c|\right)(H^c - z)s_n\right)_{\mathscr{H}^c}$ tends to zero in contradiction with the fact that $\left(1 + \sum_i |x_i^c|\right)$ is greater than one; hence $f = 0$.

In order to prove the theorem for general n_0, it is now sufficient to use a simple recurrence process and commutation relations (13) and (14); the proof that $\Delta_{n_0}^c = \bigcap_{|(n)| \leqslant n_0} \mathscr{D}\left((x^c)^{(n)}\right)$ contain $(H^c - z)\mathscr{S}^c$ as a dense subspace for each $z \in \gamma$ can be derived along the same lines as above.

3. – Analysis of two-body wave operators.

This Section is devoted to a time-dependent study of two-body reactions, that is of processes whose initial and final states involve only two bound fragments with possibly different outgoing and ingoing pairs of composite or elementary particles. Our main tool is the transitivity theorem for wave operators which is a kind of time-dependent version of the well-known distorted-wave theory; our additive interactions are obtained from the interactions V^c with the help of projection operators on bound-state wave functions. They consist of an effective one-channel interaction between bound fragments and an interaction coupling different channels. The form of wave operators obtained in this way seems to be particularly suitable for calculations and approximations.

The total interaction V^c of each cluster C will, in the first part of this Section, be supposed to satisfy one of the conditions:

A) $V^c = \sum_{i,j \in c} I^{ij}$ where I^{ij} is a two-body local potential which is real and such that $(x^{ij})^{(n)} I^{ij}(x^{ij}) \in L^2(R^3)$ for all (n) with $|(n)| \leqslant 2$.

B) $V^c = \sum_{c' \subset c} I^{c'}$ where $I^{c'}$ is a self-adjoint operator of finite rank on $\mathscr{H}^{c'}$ whose eigenvectors are in the domain of all multiplication operator by $(x^{c'})^{(n)}$, $|(n)| \leqslant 2$.

Although interactions in class *B)* have no clear physical meaning, recent works [12,13] have stressed their importance in practical or phenomenological theories of multiparticle systems as in particular for low-energy three-body

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[12] J. Gillespie: *Phys. Rev.*, **160**, 1432 (1967).
[13] A. N. Mitra: *Nucl. Phys.*, **32**, 529 (1962).

calculations based on Faddeev's formalism, where they restore the unitarity destroyed by pole-approximations on two-particle scattering amplitudes while maintaining their separability. These potentials are also used to fit the true (and unknown) nuclear interactions such as hard-core, spin-orbit or tensor forces.

Under condition *A*) or *B*) each $I^{c'}$ is $H_0^{c'}$-compact and consequently $(H_0^{c'}-\varepsilon)$-bounded; owing to theorem 1 H^c is then a self-adjoint operator with the same domain as H_0^c. Furthermore theorem 2 is satisfied for any n_0 in case *A*) and for $n_0=2$ in case *B*).

The existence proofs for wave operators are based upon two fundamental theorems of Kato. Before stating them we introduce some definitions. Let H_0 be a self-adjoint operator on $\mathscr{H}$; a linear operator V on $\mathscr{H}$ is said to be in $\mathscr{T}(H_0;\mathscr{H})$, the relative trace-class on $\mathscr{H}$ with respect to H_0, if it is $(H_0-\varepsilon)$-bounded and if for some complex z the operator $(z-H_0)^{-1}V(z-H_0)^{-1}$ is in $\mathscr{T}(\mathscr{H})$, the trace-class on $\mathscr{H}$ (*i.e.* the algebra of products of two Hilbert-Schmidt operator on $\mathscr{H}$). We shall frequently use the remark that if H_0' is another self-adjoint operator on $\mathscr{H}$ such that $\mathscr{D}(H_0)=\mathscr{D}(H_0')$ (which holds in particular if H_0-H_0' is $(H_0-\varepsilon)$-bounded) then $\mathscr{T}(H_0;\mathscr{H})=\mathscr{T}(H_0';\mathscr{H})$. Any trace-class operator on $\mathscr{H}$ obviously is in $\bigcap_H \mathscr{T}(H;\mathscr{H})=\mathscr{T}(\mathscr{H})$ where H is any self-adjoint operator on $\mathscr{H}$. More interesting examples are two-particle local interactions in $L_1(R^3)\cap L_2(R^3)$ which belong to $\mathscr{T}(H_0;L^2(R^3))$ where H_0 is the kinetic energy operator $-\Delta/2m$. Kato ([10b]) has proved the following property of perturbations in $\mathscr{T}(\mathscr{H};H_0)$:

Theorem K.1. Let H_0 be a self-adjoint operator on $\mathscr{H}$ and $V\in\mathscr{T}(H_0;\mathscr{H})$. Then H_0+V is self-adjoint and the wave operators

$$\Omega_\pm(H_0+V;H_0)=s\lim_{t\to\pm\infty}\exp[i(H_0+V)t]\exp[-H_0t]P_a^{(H_0)} \tag{15}$$

exists, $P_a^{(H_0)}$ being the projection operator on $\mathscr{M}_a(H_0)$. Furthermore these wave-operators are complete, that is

$$\Omega_\pm(H_0+V;H_0)\mathscr{M}_a(H_0)=\mathscr{M}_a(H_0+V)\,.$$

We now state the « transitivity theorem » or « chain rule » ([10c]).

Theorem K.2. Let H_0, H_0' and H be self-adjoint operator on $\mathscr{H}$ such that $\Omega_\pm(H_0';H_0)$ and $\Omega_\pm(H;H_0')$ exist. Then $\Omega_\pm(H;H_0')$ also exist and we have

$$\Omega_\pm(H;H_0)=\Omega_\pm(H;H_0')\,\Omega_\pm(H_0';H_0)\,.$$

If furthermore $\Omega_\mp(H_0';H_0)$ and $\Omega_\mp(H;H_0)$ are complete, then $\Omega_\pm(H;H_0)$ also are complete.

An immediate consequence of this theorem is:

Corollary K.2. Let H and $\{H_0^\alpha, \alpha \in I\}$ be a set of self-adjoint operators on $\mathscr{H}$ such that the wave operators $\Omega_-(H; H_0^\alpha)$ satisfy the asymptotic orthogonality condition $\Omega_\pm^*(H; H_0^\alpha)\Omega_-(H; H_0^\beta) = 0$ if $\alpha \neq \beta$ and are asymptotically complete below Λ, that is

$$\mathscr{M}_a^\Lambda(H) = \bigoplus_{\alpha \in I} \Omega_\pm(H; H_0^\alpha)\mathscr{M}_a^\Lambda(H_0^\alpha) .$$

Then if H' is a self-adjoint operator such that $\Omega_-(H'; H)$ exist and are complete, the wave operators $\{\Omega_-(H'; H_0^\alpha), \alpha \in I\}$ also exist and are asymptotically complete below Λ.

The proof of this « transitivity » property of asymptoticaly complete sets is obvious and will be omitted. We shall also need the following lemma.

Lemma 1. *Let* $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$ and P_1 (respectively P_2) be the projection operator in $\mathscr{H}$ on $\mathscr{H}_1$ (respectively $\mathscr{H}_2$). Let H_0 and $H = H_1 \oplus H_2$ be self-adjoint operators on $\mathscr{H}$ such that $P_2 P_a^{(H_0)}$ is a compact operator. Then

$$\Omega_\pm(H; H_0) = \Omega_\pm(H_1; H_0)$$

where the existence of one of these wave operators implies existence of the other.

Proof. One has $\exp[iHt] = \exp[iH_1 t]\exp[iH_2 t]$ and $\exp[iH_2 t]P_1 = P_1$; consequently:

$$\Omega_\pm(H; H_0) = s\lim_{t \to \pm\infty} (P_1 + P_2)\exp[iHt]\exp[-iH_0 t]P_a^{(H_0)} =$$

$$s\lim_{t \to \pm\infty} P_1 \exp[iH_1 t]\exp[-iH_0 t]P_a^{(H_0)} + s\lim_{t \to \pm\infty} \exp[iHt]\, P_2 P_a^{(H_0)} \exp[-iH_0 t] .$$

Since $\exp[-iH_0 t]P_a^{(H_0)}$ tends weakly to zero as t tends to $\pm\infty$, the second term vanishes in the limit by a well-known property of compact operators.

The following result of [9] is basic to our approach.

Theorem 3. Let (C_1, C_2, C_3) be a set of clusters with union $\{N\}$ and such that at least one cluster has a non-empty intersection with the two others. Then if A^{C_i}, $i = 1, 2, 3$, is a Hilbert-Schmidt operator on $\mathscr{H}^{C_i}$ the product $A^{C_1}A^{C_2}A^{C_3}$ is a Hilbert-Schmidt operator on $\mathscr{H}^{\{N\}}$.

In the following we shall write simply $\mathscr{T}(H_0)$ for $\mathscr{T}(H_0; \mathscr{H}^{\{N\}})$ and $\mathscr{T}$ for $\mathscr{T}(\mathscr{H}^{\{N\}})$. With the help of theorem 3 we can now prove the following lemma.

Lemma 2. Let the interactions satisfy conditions *A*) or *B*). Then for any two-body channel α such that $E^\alpha < \Lambda_{\{N\}}^3$ the operator $P^\alpha \Gamma^\alpha$ is in $\mathscr{T}(H^{\{N\}}) = \mathscr{T}(H^{D(\alpha)}) = \mathscr{T}(H^{\{N\}})$.

Proof. Under condition *A*) or *B*) each $I^{\{\sigma\}}$ is $(H_0^{\{N\}}\text{-}\varepsilon)$-bounded and consequently the Hilbert spaces $\mathscr{D}(H_0^{\{N\}})$, $\mathscr{D}(H^{\{N\}})$ and $\mathscr{D}(H^{D(\gamma)})$ coincide so that one has $\mathscr{T}(H_0^{\{N\}}) = \mathscr{T}(H^{\{N\}}) = \mathscr{T}(H^{D(\alpha)})$. Hence we shall prove that $P^\alpha V^\alpha \in \mathscr{T}(H^{D(\alpha)})$; since V^α and $V^{D(\alpha)}$ are $(H^{\{N\}}\text{-}\varepsilon)$-bounded, $P^\alpha V^\alpha$ is $(H^{D(\alpha)}\text{-}\varepsilon)$-bounded. Let us then see that $G_{D(\alpha)}(z) P^\alpha V^\alpha G_{D(\alpha)}(z) \in \mathscr{T}(H)$ for any complex z (here $G_{D(\alpha)}(z) = (z - H^{D(\alpha)})^{-1}$). One has

$$V = \sum_{\substack{C \cap C_i \neq \Phi \\ i=1,2}} I^C \qquad \text{for } D(\alpha) = \{C_1; C_2\} .$$

For each C such that $C \cap C_1$ and $C \cap C_2$ are nonempty we define

$$C_i' = C_i \cap C , \qquad C_i'' = C_i - C_i' . \qquad i = 1, 2 ,$$

and we analyse the contribution $P^\alpha I^C$ to $P^\alpha V^\alpha$; one has

$$P^\alpha I^C = P^\alpha \big(|x^{(C_1, C_2)}|^2 + z\big)^{-1} \big(|x^{(C_1, C_2)}|^2 + z\big) I^C , \qquad \operatorname{Im} z \neq 0 .$$

Then using the idempotence of P^α and remembering (5), we get

$$(16) \qquad G_{D(\alpha)}(z) P^\alpha I^C G_{D(\alpha)}(z) =$$
$$= \big[G_{D(\alpha)}(z) \big(|x^{(C_1, C_2)}|^2 + z\big)^{-1} P^\alpha\big] \big[P^\alpha \big(|x^{(C_1, C_2)}|^2 + z\big) I^C G_{D(\alpha)}(z)\big] .$$

It remains to show that each of these two factors is a Hilbert-Schmidt operator. For this we remark that

$$G_{D(\alpha)}(z) = (z - H_0^C)^{-1} (z - H_0^C) G_{D(\alpha)}(z)$$

and that $(z - H_0^C) G_{D(\alpha)}(z)$ is a bounded operator on $\mathscr{H}^{\{N\}}$ since

$$\mathscr{D}(H_0^C) = \mathscr{D}(H_0^{\{N\}}) = \mathscr{D}(H^{D(\alpha)}) ;$$

a similar result holds if H_0^C is replaced by $|p^{(C_1, C_2)}|^2 / 2m^{(C_1, C_2)}$ (see (1)). On the other hand we observe that if $V(x)$ is the operator of multiplication by a square integrable function on R^3 in the representation space of a three-dimensional system, then its product with the resolvent operator $(H_0 + z)^{-1}$ where H_0 is the kinetic-energy operator of this system, is a Hilbert-Schmidt operator due to the fact that the kernel of this resolvent operator is proportional to the Green's

function $(\exp[-i\sqrt{z}|x-y|])/|x-y|$. Consequently $\left((|p^{(C_1,C_2)}|^2/2m^{(C_1,C_2)})-z\right)^{-1}$ is a Hilbert-Schmidt operator on $\mathscr{H}^{(C_1,C_2)}$ and owing to (5) $G_{D(\alpha)}(z)(|x^{(C_1,C_2)}|^2+z)^{-1}P^\alpha$ is a Hilbert-Schmidt operator on $\mathscr{H}^{\{N\}}$. Let us now decompose $x^{(C_1,C_2)}$ as

$$x^{(C_1,C_2)} = n^{(C_1',C_1'')}x^{(C_1',C_1'')} + n^{(C_2',C_2'')}x^{(C_2',C_2'')} + x^{(C_1',C_2')},$$

where $n^{(C_i',C_i'')}=m_{C_i'}/m_{C_i}$; then if we use the fact that by theorem 2 (see also lemma 3 below) the eigenvectors of P^α are in the domains of $|x^{(C_i',C_i'')}|^2$, we see by theorem 3 that the second factor in (16) also is in the Hilbert-Schmidt class on $\mathscr{H}^{\{N\}}$; this is immediate under assumption B) while in the case A) it comes from the fact that for any two-particle cluster the operator $(1+|x^{ij}|^2)\cdot I^{ij}(z-H_0^{ij})^{-1}$ is in the Hilbert-Schmidt class on $\mathscr{H}^{ij}$. Finally $P^\alpha\Gamma^C$ being in $\mathscr{T}(H_0^{\{N\}}+V^{D(\alpha)})$ for each component Γ^C of Γ^α, the operator $P^\alpha V^\alpha$ also is (*).

We now have all the tools required for the proof of theorem 4.

Theorem 4. Under conditions A) or B) on the interactions, the wave operators $\Omega_\pm^\alpha$ exist for any two-body channel α with $E^\alpha<\Lambda_{\{N\}}^3$ and they are given by

$$\Omega_\pm^\alpha = \Omega_\pm(H^{\{N\}};\ P^\alpha H^{\{N\}}P^\alpha + Q^\alpha H^{\{N\}}Q^\alpha)\cdot {}^{sc}\Omega_\pm^\alpha \tag{17}$$

the single-channel operator ${}^{sc}\Omega_\pm^\alpha$ being defined by

$${}^{sc}\Omega_\pm^\alpha = \Omega_\pm\left(\frac{|p^{(C_1,C_2)}|^2}{2m^{(C_1,C_2)}}+V_{sc}^\alpha;\frac{|p^{(C_1,C_2)}|^2}{2m^{(C_1,C_2)}}\right)P^\alpha, \tag{18}$$

where $D(\alpha)=\{C_1;C_2\}$ and the « semi-classical » interactions V_α^{sc} in channel α is defined by

$$V_{sc}^\alpha P^\alpha = P^\alpha V^\alpha P^\alpha .$$

Furthermore the wave operators $\Omega_\pm(H^{\{N\}};P^\alpha H^{\{N\}}P^\alpha\oplus Q^\alpha H^{\{N\}}Q^\alpha)$ and

$$\Omega_\pm\left(\frac{|p^{(C_1,C_2)}|^2}{2m^{(C_1,C_2)}}+V_{sc}^\alpha;\frac{|p^{(C_1,C_2)}|^2}{2m^{(C_1,C_2)}}\right),$$

are complete.

Proof. One has

$$H^{\{N\}} = P^\alpha H^{\{N\}}P^\alpha + Q^\alpha H^{\{N\}}Q^\alpha + P^\alpha V^\alpha Q^\alpha + Q^\alpha V^\alpha P^\alpha .$$

(*) It has to be noticed that in case B) if the eigenvectors of each Γ^C are in $\mathscr{D}_2^0$ then $P^\alpha V^\alpha$ is in fact in the trace-class on $\mathscr{H}^{\{N\}}$. This results from theorems of [9] concerning products of many-particle trace-class operators.

By lemma 2 and theorem K.1 the wave operators

$$\Omega_{\pm}(H^{\{N\}};\ P^{\alpha}H^{\{N\}}P^{\alpha} + Q^{\alpha}H^{\{N\}}Q^{\alpha})$$

exist and are complete. Let us then consider

$$\Omega_{\pm}(P^{\alpha}H^{\{N\}}P^{\alpha} + Q^{\alpha}H^{\{N\}}Q^{\alpha};\ H_0^{\alpha})\ ;$$

by lemma 1 they are equal to $\Omega_{\pm}(P^{\alpha}H^{\{N\}}P^{\alpha};\ H_0^{\alpha})$ where one has

$$P^{\alpha}H^{\{N\}}P^{\alpha} = H_0^{\alpha} + P^{\alpha}V^{\alpha}P^{\alpha}\ .$$

Now $P^{\alpha}V^{\alpha}P^{\alpha}$ defines a linear operator on $P^{\alpha}\mathscr{H}^{\{N\}} = \mathscr{H}^{(C_1,C_2)} \otimes \varphi^{b(C_1)} \otimes \varphi^{b(C_2)}$ and consequently a linear operator V_{sc}^{α} on $\mathscr{H}^{(C_1,C_2)}$. By lemma 2 the operator

$$P^{\alpha}(z - H_0^{\alpha})^{-1}V_{sc}^{\alpha}(z - H_0^{\alpha})^{-1} = G_{D(\alpha)}(z)P^{\alpha}V^{\alpha}P^{\alpha}G_{D(\alpha)}(z)$$

is in $\mathscr{T}$; consequently

$$V_{sc}^{\alpha} \in \mathscr{T}\left(\frac{|p^{(C_1,C_2)}|^2}{2m^{(C_1,C_2)}};\ \mathscr{H}^{(C_1,C_2)}\right),$$

by (5). Since

$$P^{\alpha}H^{\{N\}}P^{\alpha} = \left[\frac{|p^{(C_1,C_2)}|^2}{2m^{(C_1,C_2)}} + E^{\alpha} + V_{sc}^{\alpha}\right]P^{\alpha}$$

and since the constant E^{α} does not play any role in the definition of

$$\Omega_{\pm}\left(\frac{|P^{(C_1,C_2)}|^2}{2m^{(C_1,C_2)}} + E^{\alpha} + V_s^{\alpha};\ \frac{|P^{(C_1,C_2)}|^2}{2m^{(C_1,C_2)}} + E^{\alpha}\right),$$

as a time limit (15) one has then

$$\Omega_{\pm}(P^{\alpha}H^{\{N\}}P^{\alpha};\ H_0^{\alpha}) = \Omega_{\pm}\left(\frac{|P^{(C_1,C_2)}|^2}{2m^{(C_1,C_2)}} + V_{sc}^{\alpha};\ \frac{|P^{(C_1,C_2)}|^2}{2m^{(C_1,C_2)}}\right)P^{\alpha}\ ,$$

where the wave operators on the right-hand side exist and are complete. Now (17) follows from the transitivity theorem K.2 and this concludes the proof of theorem 4.

Before studying more general interactions, let us discuss theorem 4. In the expression (17) the wave operator ${}^{sc}\Omega_{\pm}^{\alpha}$ describes the elastic scattering of two stable particles with masses m_{C_1} and m_{C_2} interacting through the two-body

potential V_{sc}^{α}; if the interactions $V^{(c)}$ are local, V_{sc}^{α} can be shown (Appendix IV) to represent the mean interaction due to some mass distributions. As to the wave operators $\Omega_{\pm}(H^{(N)}; P^{\alpha}H^{(N)}P^{\alpha} + Q^{\alpha}H^{(N)}Q^{\alpha})$ they describe the effects due to the coupling between channels.

This coupling is expected to be important only in finite regions of the configuration space: intuitively we should say that this is implied by the « asymptotic stability » of the bound states that we consider, resulting on the one hand from the fact that the bound state energies $E^{b(\sigma)}$ are not limit points of the spectrum of H^c, and in particular are not embedded in the continuum, so that a certain amount of energy is necessary to break them up; and on the other hand from the fast asymptotic decrease of the bound state wave functions as expressed by theorem 2 which asserts that when the incident or emergent bound states are sufficiently far from each other then their constituents also are, so that the interaction between them is not strong enough to produce break-up and rearrangement processes. That intuitive reasoning can be improved by performing asymptotic estimates on the kernel of $P^{\alpha}V^{\alpha}Q^{\alpha} + \text{h.c.}$ in configuration space; such estimates are easily obtained if the asymptotic behaviour of the I^{ij} is known and confirm rigorously the short-range nature of rearrangement forces. Thus (17) gives a very simple picture of the scattering process according to which the two ingoing or outgoing bound states first interact through V_{sc}^{α} before the inelastic processes occur. Furthermore, they suggest that under the « asymptotic stability » condition for bound states described above, (17) remains valid for a wider class of interactions such as for example local two-particle interactions I^{ij} in $L^2(R^3)+L^P(R^3)$, $2 \leqslant P < 3$ which have been studied by HACK ([14]). For such systems the existence of ${}^{sc}\Omega_{\alpha}^{\pm}$ results from the fact that if the I^{ij} are in Hack's class, the V_{sc}^{α} also are as follows from results of Appendix (II.2) and Holder's inequality; however we do not know whether they satisfy the completeness condition except if the I^{ij} (and consequently the V_{sc}^{α}) are rotationally invariant in which case completeness holds ([15]). As to the $\Omega_{+}(H^{(N)}; P^{\alpha}H^{(N)}P^{\alpha} + Q^{\alpha}H^{(N)}Q^{\alpha})$ they can be expected to exist and be complete since these wave operators arise from short-range forces.

If there are Coulomb admixtures a special treatment is needed but we shall see that a form similar to (17) can be obtained for Dollard's « distorted »-wave operators ([16]). DOLLARD has shown that these « distorted »-wave operators exist, provided bound-state wave functions are in the domain of the same power of $|x^c|$, which holds owing to theorem 2 under the three-body threshold.

([14]) M. N. HACK: *Nuovo Cimento*, **13**, 231 (1959).
([15]) S. T. KURODA: *Nuovo Cimento*, **12**, 431 (1959).
([16]) J. D. DOLLARD: *Journ. Math. Phys.*, **5**, 729 (1964).

They are given by

$$W_{\pm}^{\alpha} = s \lim_{t \to \pm\infty} \exp[iH_{\sigma}^{\{N\}}t] \exp[-iH_{0\sigma}^{\alpha}t]P^{\alpha} .$$

Here $H_{\sigma}^{\{N\}} = H_{0}^{\{N\}} + \sum_{i<j} (e_i e_j/|x^{ij}| + I^{ij})$, where I^{ij} is square integrable, and $H_{0\sigma}^{\alpha}(t)$ is the one-parameter family of self-adjoint operators defined for $D(\alpha) = \{C_1; C_2\}$ by

$$H_{0\sigma}^{\alpha}(t) = H_{0}^{(C_1,C_2)} + \frac{\varepsilon(t)\, m^{(C_1,C_2)}}{|p^{(C_1,C_2)}|}\, e_{C_1} e_{C_2} \log\left(2|t|H_{0}^{(C_1,C_2)}\right) ;$$

here e_C is the total charge of cluster C and $\varepsilon(t) = \pm 1$ for $t \gtrless 0$. Furthermore if $N = 2$ and the interaction is purely Coulomb then $W_{\pm}$ is given by

$$W_{\pm} = \int \mathrm{d}p |\psi_{p}^{\pm})(p| , \tag{19}$$

where $|p)$ denotes a plane wave with momentum p and $|\psi_{p}^{\pm})$ are the stationary scattering solutions of the Coulomb potential with outgoing or ingoing momentum equal to p. The physical meaning of these « wave operators » is that although the wave packets

$$f_t(p^{(C_1,C_2)}) = (P^{\alpha} \exp[-iH_{\sigma}^{\{N\}}t]f)(p^{(C_1,C_2)})$$

do not behave asymptotically like free wave packets, the momentum distributions $|f_t(p^{(C_1,C_2)})|^2$ tend to the asymptotic limits $|(W_{\pm}^{\alpha *} f)(p^{(C_1,C_2)})|^2$ which are in fact the only things needed for the computation of cross-sections. However for this last purpose one needs a practicable from of $W_{\pm}^{\alpha}$; for this we split the interaction V_{sc}^{α} into the sum

$$V_{sc}^{\alpha} = V_{sc}^{\alpha s} + \frac{e_{C_1} e_{C_2}}{|x^{(C_1,C_2)}|} ,$$

where the « short-range » part $V_{sc}^{\alpha s}$ comes from the contribution to V_{sc}^{α} of the I^{ij} and of the « multipole » moments of the charge distributions $|\varphi^{b(C)}(x^C)|^2$, which are finite by theorem 2 since the interactions is purely local, and give contributions in $O(1/|x^{(C_1,C_2)}|^2)$ to V_{sc}^{α}. We now use the observation that Kato's proof of the transitivity theorem K.2 remains valid if one of the wave operators in theorem K.2, say $\Omega_{\pm}(H_0'; H_0)$, is replaced by a distorted wave operator intertwining the absolutely continuous parts of H_0 and H_0'. Using that extension of theorem K.2 we can prove the existence of

$$\Omega_{\pm}\left(H_{0}^{(C_1,C_2)} + V_{sc}^{\alpha};\ H_{0}^{(C_1,C_2)} + \frac{e_{C_1} e_{C_2}}{|x^{(C_1,C_2)}|}\right) ,$$

and finally the equality

$$(17a) \quad W_{\pm}^{\alpha} = \Omega_{\pm}(H^{\{N\}}; P^{\alpha} H^{\{N\}} P^{\alpha} + Q^{\alpha} H^{\{N\}} Q^{\alpha}) \cdot \cdot \Omega_{\pm}\left(H_0^{(C_1,C_2)} + V_{sc}^{\alpha}; H_0^{(C_1,C_2)} + \frac{e_{C_1} e_{C_2}}{|x^{(C_1,C_2)}|}\right) \cdot W_{\pm}^{(C_1,C_2)} ,$$

where $W_{\pm}^{(C_1,C_2)}$ is Dollard's wave operator (19) for the « particles » C_1 and C_2. The interest of (17a) lies in the fact that the long-range effects of the Coulomb interaction appear only in $W_{\pm}^{(C_1,C_2)}$.

4. — Unitarity of the S-matrix below the three-body threshold.

In this Section we restrict ourselves to potentials of type A) or B). As we have seen in Sect. 1, the S-matrix is unitary under $\Lambda_{\{N\}}^3$ if we can prove the asymptotic completeness relation

$$(20) \qquad \mathcal{M}_a^{(\Lambda_{\{N\}}^3)}(H^{\{N\}}) = \bigoplus_{\alpha} \Omega_{\pm}^{\alpha} \mathcal{M}_a^{(\Lambda_{\{N\}}^3)}(H_0^{\alpha}) .$$

Under $\Lambda_{\{N\}}^3$ the only channels involved in (20) are two-body channels and in order to prove this relation it is sufficient to show that it is true for every $\Lambda < \Lambda_{\{N\}}^3$. The advantage of replacing $\Lambda_{\{N\}}^3$ by Λ results from

Lemma 3. For every $\Lambda < \Lambda_{\{N\}}^3$ there is only a finite number of channels α with $E^{\alpha} \leqslant \Lambda$; if $\alpha = \{b(C_1), b(C_2)\}$ satisfies this condition, then $E^{b(C_i)} < \Lambda_{C_i}^2$, $i = 1, 2$.

Proof. Let us suppose that there exists an infinity of channel energies E^{α} in the interval $[\Lambda_{\{N\}}^2, \Lambda]$. This implies the existence of an infinite sequence α_n with $D(\alpha_n) = \{C_1; C_2\}$ for all n and such that $E^{b_n(C_1)} + E^{b_n(C_2)}$ has a limit point $\bar{E}_1 + \bar{E}_2 \leqslant \Lambda$; then at least $\bar{E}_1$ is the limit of an infinite sequence $E^{b_n(C_1)}$. By theorem 1 we then have $\bar{E}_1 \geqslant \Lambda_{C_1}^2$. But $\Lambda_{C_1}^2$ is equal to some $E^{b(C_1')} + E^{b(C_1'')}$ with $C_1' \cap C_1'' = C_1$ and this contradicts $\bar{E}_1 + \bar{E}_2 \leqslant \Lambda < \Lambda_{\{N\}}^3$.

We now define for $\Lambda < \Lambda_{\{N\}}^3$

$$B^{\Lambda}(D) = \{\alpha | D(\alpha) = D \text{ and } E^{\alpha} \leqslant \Lambda\} .$$

The set of (two-fragment) cluster decompositions with nonempty $B^{\Lambda}(D)$ are denoted by $D_1, \ldots, D_n$; each $B^{\Lambda}(D)$ contains a finite number of channels and (20) holds if for all $\Lambda < \Lambda_{\{N\}}^3$ one has

$$(20a) \qquad \mathcal{M}_a^{\Lambda}(H^{\{N\}}) = \bigoplus_{k=1}^{n} \Big(\bigoplus_{\alpha \in B^{\Lambda}(D_k)} \Omega_{\pm}^{\alpha} \mathcal{M}_a^{\Lambda}(H_0^{\alpha}) \Big) .$$

This relation is obvious if no D_k exists (*i.e.* if $\Lambda \leqslant \Lambda^2_{\{N\}}$), and results then from theorem 1. In the general situation we define

$$H^\Lambda = H^{\{N\}} - \sum_{k=1}^{n} \sum_{\alpha \in B^\Lambda(D_k)} (E^\alpha - \Lambda) P^\alpha .$$

By theorem K.0, the operator H^Λ is self-adjoint, and $\mathscr{D}(H^\Lambda) = \mathscr{D}(H_0^{\{N\}})$; it satisfies furthermore the following lemma.

Lemma 4. The continuous spectrum of H^Λ is contained in $[\Lambda, +\infty[$.

The proof of this lemma requires the techniques used in ([11]). We refer to this paper for more complete details of the arguments that we give now and in particular for the derivation of the Weinberg equation by establishing a one-to-one correspondence between the terms of the perturbation series for the resolvent operator $(z - H^\Lambda)^{-1}$ and graphs; in ([11]) only interactions $I^{\{\sigma\}}$ are considered and in order to take into account the new type of perturbations $(E^\alpha - \Lambda) P^\alpha$ with $P^\alpha = P^{b(\sigma_1)} P^{b(\sigma_2)}$ an additional convention is needed which consists in representing them by two vertical connections standing on the same vertical line, the one between particles of C_1 and the other between particles of C_2. In the recombination of the terms of the infinite perturbation series according to the connectivity of their associated graph, one has then to consider more general types of increasing sequences of cluster decompositions $S = (D_0, D_1, \ldots, D_k)$; for example two consecutive terms in S can be

$$D_i = \{C_1; C_2, \ldots, C_n\} \quad \text{and} \quad D_{i+1} = \left\{ \bigcup_{i=1}^{p} C_i; \bigcup_{j=p}^{n} C_j \right\} .$$

One obtains in this way the Weinberg equation

$$(z - H^\Lambda)^{-1} = D^\Lambda(z) + I^\Lambda(z)(z - H^\Lambda)^{-1} .$$

$D^\Lambda(z)$ and $I^\Lambda(z)$ depend on z through the resolvents

$$G_D^\Lambda(z) = \left(z - \left(H^D - \sum_{\alpha \in B^\Lambda(D)} (E^\alpha - \Lambda) P^\alpha \right) \right)^{-1} .$$

So that $D^\Lambda(z)$ and $I^\Lambda(z)$ are analytic in the complex plane cut along $[\Lambda, +\infty[$; the proof of the compactness of $I^\Lambda(z)$ in this open domain requires no modification due to the fact that the $P^{b(\sigma)}$ are compact operators on $\mathscr{H}^{\{\sigma\}}$; one derives then the analyticity of $(z - H^\Lambda)^{-1}$ in this domain by standard arguments.

It results from this lemma that (20a) holds if $H^{\{N\}}$ is replaced by H^Λ. We start from this artificial situation and restore successively their original place to the channel energies E^α, $\alpha \in B^\Lambda(D_k)$, $k = 1, 2, \ldots, n$, which have been shifted

to A in H^A. For this we define

$$H_k^A = H^A + \sum_{i \leqslant k} \sum_{\alpha \in B^A(D_i)}^{\oplus} (E^\alpha - A) P^\alpha ,$$

and make the following recurrence hypothesis:

$$(R_k) \qquad \mathscr{M}_a^A(H_k^A) = \bigoplus_{\substack{\alpha \in B^A(D_i) \\ i \leqslant k}} \Omega_\pm(H_k^A; H_0^\alpha) \mathscr{M}_a^A(H_0^\alpha) .$$

The existence of $\Omega_\pm(H_k^A; H_0^\alpha)$, $\alpha \in B(D_i)$, $i \leqslant k$, results from theorem K.2 and the first part of the following lemma.

Lemma 5. Let

$$P_k^A = \sum_{\alpha \in B^A(D_k)}^{\oplus} P^\alpha , \qquad Q_k^A = I - P_k^A \qquad \text{and} \qquad h_k^A = P_k^A H_k^A P_k^A + Q_k^A H_k^A Q_k^A .$$

Then the wave operators $\Omega_\pm(H_k^A; h_k^A)$ and $\Omega_\pm(h_k^A; H_0^\alpha)$ exist for all $k = 1, 2, \ldots, n$ and $\alpha \in B^A(D_i)$, $i \leqslant k$. Furthermore

i) $\Omega_\pm(H_k^A; h_k^A)$ are complete

ii) $\bigoplus_{\alpha \in B^A(D_k)} \Omega_\pm(h_k^A; H_0^\alpha) \mathscr{M}_a^A(H_0^\alpha) = \mathscr{M}_a^A(P_k^A H_k^A P_k^A)$.

Proof. One has

$$(21) \qquad H_k^A - h_k^A = P_k^A \Big([V^{\{N\}} - V^{D_k}] - \sum_{\substack{\alpha \in B^A(D_i) \\ i \geqslant k+1}} (E^\alpha - A) P^\alpha \Big) Q_k^A + \text{h.c.}$$

As results from the remark following lemma 2, $P_k^A P^\alpha$ and its Hermitian conjugate are in $\mathscr{T}$ (hence in $\mathscr{T}(h_k^A)$) for any two-body channel α with $D(\alpha) \neq D_k$; as to the term $P_k^A [V^{\{N\}} - V^{D_k}] Q_k^A + \text{h.c.}$ it is in $\mathscr{T}(H^{D_k})$ by lemma 2; but it is easily seen that $\mathscr{T}(H^{D_k}) = \mathscr{T}(H_k^A) = \mathscr{T}(h_k^A)$ so that $\Omega_\pm(H_k^A; h_k^A)$ exist and are complete by theorem K.1.

A similar analysis applied to

$$P_k^A H_k^A P_k^A - H^{D_k} = P_k^A \Big([V^{\{N\}} - V^{D_k}] - \sum_{\substack{\alpha \in B^A(D_i) \\ i \geqslant k+1}} (E^\alpha - A) P^\alpha \Big) P_k^A$$

leads to the result that $h_k^A - H^{D_k} \in \mathscr{T}(H^{D_k})$. On the other hand by lemma 1 one has $\Omega_\pm(h_k^A; H_0^\alpha) = \Omega_\pm(P_k^A H_k^A P_k^A; H^{D_k}) P^\alpha$ for $D(\alpha) = D_k$; since $\mathscr{M}_a^A(H^{D_k}) = \bigoplus_{\alpha \in B^A(D_k)} \mathscr{M}_a^A(H_0^\alpha)$, another application of theorem K.1 gives ii).

It results from i) that (R) is equivalent to

$$(R_k') \qquad \mathscr{M}_a^A(h_k^A) = \bigoplus_{\substack{\alpha \in B^A(D_i) \\ i \leqslant k}} \Omega_\pm(h_k;\, H_0)\mathscr{M}_a^A(H_0^\alpha)\,.$$

Let us prove that (R_{k+1}') holds if (R_k) is true and for this we first effect in the right member of (R_{k+1}') the summation over the channels $\alpha \in B^A(D_{k+1})$; owing to ii) of lemma 5 this gives $\mathscr{M}_a^A(P_{k+1}^A H_{k+1}^A P_{k+1}^A)$. So it remains to show that

$$(22) \qquad \bigoplus_{\substack{\alpha \in B^A(D_i) \\ i \leqslant k}} \Omega_\pm(h_{k+1}^A;\, H_0^\alpha)\mathscr{M}_a^A(H_0^\alpha) = \mathscr{M}_a^A(Q_{k+1}^A H_{k+1}^A Q_{k+1}^A)\,.$$

We use then the relation

$$(23) \qquad Q_{k+1}^A(H_{k+1}^A - H_k^A) = 0\,,$$

which together with lemma 1 gives

$$\Omega_\pm(h_{k+1}^A;\, H_0^\alpha) = \Omega_\pm(P_{k+1}^A H_k^A P_{k+1}^A + Q_{k+1}^A H_k^A Q_{k+1}^A;\, H_k^A)\cdot\Omega_\pm(H_k^A;\, H_0^\alpha)\,.$$

Existence and completeness of $\Omega_\pm(P_{k+1}^A H_k^A P_{k+1}^A + Q_{k+1}^A H_k^A Q_{k+1}^A;\, H_k^A)$ results from theorem K.1; in fact (23) implies that these wave operators arise from the perturbation (21) with $k+1$ instead of k and which is then in $\mathscr{T}(H_{k+1}^A) = \mathscr{T}(H_k^A)$. So (R_k), corollary K.2 and lemma 1 give (22). It now remains to remark that (R_n) is simply $(20a)$ to conclude.

5. – Three-particle systems.

As an illustration of the new features appearing in a time-dependent approach to three-body collisions we now study the S-matrix completely in the simplest situation $N = 3$. The philosophy of our analysis of three-particle wave-operators is similar to that of Faddeev's work [6] in which the two-body problems are supposed to be solved and their solutions used in the calculation of three-body effects; these three-body effects lead in our approach to « small perturbation » problems which, we expect, can be treated by perturbation techniques.

For convenience we slightly modify the notations previously used. Anything related to the relative motion of a two-particle cluster ij and the third particle k will be indexed by (k) instead of (ij, k). A two-body channel $\alpha = (b(ij), k)$ will be simply designated by $b(ij)$ and the index f will refer to the channel in which all particles are free. The total kinetic-energy operator will

be designated simply by H_0 and owing to (1) we have

$$H_0 = H_0^{ij} + H_0^{(k)} .$$

We denote by $H = H_0 + V$ the total energy operator and assume the total interaction V to be equal to $V^{12} + V^{23}$; the interaction V^{13} has been set equal to zero for simplicity. The main result to be proved in this Section is the following.

Theorem 5. Assume that the two-body potentials V^{ij} satisfy condition A) or B) and furthermore that the singular subspace (see Appendix I) of each two-body Hamiltonian $H^{ij} = H_0^{ij} + V^{ij}$ is spanned by a finite number of boundstates $b(ij)$ with $E^{b(ij)} < 0$. Let

$$\Omega_{\pm}^{ij} = \Omega_{\pm}(H^{ij}; H_0^{ij}) \quad \text{and} \quad P_a^{ij} = \Omega_{\pm}^{ij}\Omega_{\pm}^{ij*} , \quad Q_a^{ij} = 1 - P_a^{ij} .$$

Then

i) the two-body wave operators $\Omega_{\pm}^{b(ij)}$ exist and are given by (17).

ii) The free-channel wave operators $\Omega_{\pm}^{f} = \Omega_{\pm}(H; H_0)$ exist if and only if

$$(\Omega_{12}^{23})_{\pm} = \Omega_{+}(H_0 + \Omega_{\pm}^{12*} V^{23} \Omega_{\pm}^{12}; H_0 + V^{23})$$

exist; one has then (*)

$$\Omega_{\pm}^{f} = \Omega_{\pm}(H; P_a^{12} H P_a^{12} + Q_a^{12} H Q_a^{12}) \cdot \Omega_{\pm}^{12} \cdot (\Omega_{12}^{23})_{\pm} \cdot \Omega_{\pm}^{23} . \tag{24}$$

iii) The asymptotic completeness (11) holds for every A if and only if the wave operators (Ω_{12}^{23}) are complete.

Before proving this theorem we need the following lemmas.

Lemma 6. Under assumption A) or B) the wave operators $\Omega_{\pm}^{ij}$ exist, are complete and satisfy $\Omega_{\pm}^{ij} = \Omega_{\pm}(H_0 + V^{ij}; H_0)$.

Proof. The potentials V^{ij} are in $\mathscr{T}(H_0^{ij}; \mathscr{H}^{ij})$ so existence and completeness of $\Omega_{\pm}^{ij}$ result from theorem K.1; definition (15) and the fact that $H_0^{(k)}$ and V^{ij} commute with each other for $(ijk) = (123)$ complete the proof.

Let us remark that the wave operators $\Omega_{\pm}(H_0 + V^{ij}; H_0)$ are complete if and only if $\mathscr{M}_a(H^{ij}) = \mathscr{H}^{ij}$ as results from theorem A.1.

(*) This formula is an extension of a result obtained by COESTER ([17]) in the one-channel case.

([17]) F. COESTER: private communication.

Lemma 7. Under the conditions stated at the beginning of theorem 5 one has

$$s \lim_{t\to\pm\infty} (\Omega_{\pm}^{ij} - 1) \exp[-i(H_0 + V^{jk})t] = 0 \tag{25}$$

for $(ijk) = (123)$.

Proof. We use the decomposition of the identity operator (AI.6)

$$1 = P_a^{jk} + \sum_{b(jk)} P^{b(jk)} \tag{26}$$

resulting from $P_a^{jk} = P_a^{(H^{jk})}$.

It follows from lemma 6 that

$$s \lim_{t\to\pm\infty} (\Omega_{\pm}^{ij} - 1) \exp[-i(H_0 + V^{jk})t] P_a^{jk} = s \lim_{t\to\pm\infty} (\Omega_{\pm}^{ij} - 1) \exp[-iH_0 t] \Omega_{\pm}^{jk*}$$

and the last limit is easily seen to be zero. It remains to show that

$$s \lim_{t\to\pm\infty} (\Omega_{\pm}^{ij} - 1) \exp[-iH_0^{(k)} t] P^{b(jk)} = 0 \ ;$$

although this result is intuitively clear, the only proof that we know is rather long and will be given in Appendix III.

We now return to the proof of theorem 5. Statement i) results simply from theorem 4. Let us then show the sufficiency of the condition stated in ii) for the existence of $\Omega_{\pm}'$. Owing to the assumptions of theorem 5 and to (26) one has

$$Q_a^{12} = \sum_{b(ij)} P^{b(12)}$$

where the sum is finite and the $P^{b(ij)}$ are projection operators over isolated bound-states wave functions; consequently the eigenfunctions of Q_a^{12} are in $\mathscr{Q}_2^{12}$ by theorem 2; then the wave operators $\Omega_{\pm}(H; P_a^{12} H Q_a^{12} + Q_a^{12} H Q_a^{12})$ exist and are complete by lemma 2 and theorem K.1. Now using $P_a^{12} = \Omega_{\pm}^{12} \Omega_{\pm}^{12*}$ and operating on the left of (25) with Q_a^{12} and $\Omega_{\pm}^{12*}$ we get with the help of lemma 7

$$\Omega_{+}(P_a^{12} H P_a^{12} + Q_a^{12} H Q_a^{12}; H_0 + V^{23}) \quad \Omega_{\pm}(P_a^{12} H P_a^{12}; H_0 + V^{23}) \quad \Omega_{\pm}^{12}(\Omega_{12}^{23})_{-} \ , \tag{27}$$

the existence of one term in this equality implying existence of the other. Using the transitivity theorem and lemma 6 we then get relation (24).

Now the necessity of the condition in ii) is again a consequence of the existence of $\Omega_{\pm}(P_a^{12} H P_a^{12} + Q_a^{12} H Q_a^{12}; H)$ and the transitivity theorem; the proof is simple and will be omitted.

If the assumption $V^{13}=0$ is removed and V^{13} satisfies the conditions stated in theorem 5 for V^{12} and V^{23} we must require furthermore the existence of $\Omega_{\pm}(H_0+\Omega_{\pm}^{13*}(V^{23}+V^{12})\Omega_{\pm}^{13}; H_0+V^{23}+V^{12})$; this can be shown along the same lines as above and leads to a relation similar to (24) for $\Omega_{\pm}^{f}$ but involving also the above wave operator and $\Omega_{\pm}^{13}$.

In order to prove iii) we first suppose that $(\Omega_{12}^{23})_{-}$ are complete; it is immediately seen with the help of lemma 2 and theorem K.1 that

$$\Omega_{\pm}\Big(H; P_a^{12}HP_a^{12}+\Big(\sum_{b(12)}P^{b(12)}HP^{b(12)}\Big)\Big)$$

also are complete and

$$\mathscr{M}_a(P^{b(12)}HP^{b(12)})=\Omega_{\pm}(P^{b(12)}HP^{b(12)}; H_0^{b(12)})\mathscr{M}_a(H^{b(12)})$$

(see the proof of theorem 4); it results from corollary K.2 and lemma 1 that (11) for all A is equivalent to

$$\sum_{\beta\neq b(12)}\Omega_{\pm}(P_a^{12}HP_a^{12}; H_0^{\beta})\mathscr{M}_a(H_0^{\beta})=\mathscr{M}_a(P_a^{12}HP_a^{12})\,.$$

By corollary K.2 and (27) this is equivalent to

$$\sum_{\beta\neq b(12)}\Omega_{\pm}(H_0+V^{23}; H_0^{\beta})\mathscr{M}_a(H_0^{\beta})=\mathscr{M}_a(H_{\cdot}+V^{23})\,.$$

But the remaining summation is over all bound states $b(23)$ and the channel f for which $H_0^{\beta}=H_0$; by lemma 6 the above equality is then simply

$$P_a^{23}+\sum_{b(23)}P^{b(23)}=1\,.$$

since by the theorem AI iii) the projection operator on the absolutely continuous part of H_0+V^{23} is the identity. But this equality is satisfied under the assumption of theorem 5 and this completes that part of the proof.

Now reversing the order of the proof one can see that the condition ii) also is necessary for the asymptotic completeness to hold.

Let us now discuss theorem 5. First of all one must notice that under the conditions stated, the wave operators $\Omega_{\pm}^{f}$ exist; in fact interactions of type A) belong to Hack's class ([14]) and COESTER ([18]) has given an existence proof for interactions of type B); the introduction of $(\Omega_{12}^{23})_{-}$ is mainly motivated by the

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([18]) F. COESTER: *Helv. Phys. Acta*, **38**, 7 (1965).

remaining problems which are first to get integral equations for scattering states without singular kernels and second to prove asymptotic completeness. But precisely $(\Omega_{12}^{23})_{\pm}$ arise from the perturbations of $H_0 + V^{23}$ by

$$A_{\pm}^{12} V^{23} = \Omega_{\pm}^{12*} V^{23} \Omega_{\pm}^{12} - V^{23} .$$

An important property of this operator is derived from Ekstein's lemma (Appendix II) and is its asymptotic decrease in any direction of the configuration space; namely if f is a vector in $\mathscr{D}(H_0)$ then

$$s \lim_{|R| \to \pm\infty} A_{\pm}^{12} V^{23} \mathscr{U}(R) f = 0 , \tag{28}$$

where R is a six-dimensional vector in the configuration space of the system and $\mathscr{U}(R)$ the corresponding translation operator. This is the reason why $A_{\pm}^{12} V^{23}$ are much better than, say, V^{23}; (28) is the first quality that a perturbation must satisfy in order to be « small », that is to produce no enlargment of the spectrum and the multiplicity of the perturbed Hamiltonian. One could even expect that for interactions of type B) $A_{\pm}^{12} V^{23}$ are in $\mathscr{T}$; unfortunately it can be proved that this does not hold except for a very particular class of potentials such that $I - \Omega_{\pm}^{12}$ is separable (but for these systems $S^{12} - \Omega_{+}^{12*} \Omega_{-}^{12} - 1$). However our hope is that (28) is sufficient to prove completeness of $(\Omega_{12}^{23})_{\pm}$.

Let us now discuss the first conditions of theorem 5. The requirement that there exists only a finite number of bound states with $E^{b(ij)} < 0$ is rather weak and holds in fact for potentials of type A) ([19]) (however for potentials of type B) bound states embedded in the continuum can appear). A more important requirement is that the subspaces $\mathscr{M}_{cs}(H^{ij})$ are empty; their occurrence generally breaks the unitarity of the S-matrix. This comes from the fact that $\mathscr{M}_{cs}(H^{ij})$ has no reason to reduce the operator

$$S(H; H_0 + V^{ij}) = \Omega_{+}^{*}(H; H_0 + V^{ij}) \Omega_{-}(H; H_0 + V^{ij})$$

(a precise example is given in Appendix V); as a consequence there exist final states having the form, say, $\exp[-iH_0^{(1)}t] f^1 \otimes \exp[-H^{23}t] f^{23}$ where $f^{23} \in \mathscr{M}_{cs}(H^{23})$. Such states do not behave asymptotically like free-wave packets in any of the channel subspace. Although RUELLE ([20]) has shown that

$$\lim_{T \to \pm\infty} \frac{1}{T} \int_0^T \mathrm{d}t \int_{|x^{23}| < R} \mathrm{d}x^{23} |(\exp[-iH^{23}t] f^{23})(x^{23})|^2 = 0 ,$$

([19]) J. SCHWINGER: *Proc. Nat. Acad. Sci.*, **47**, 122 (1961).

([20]) D. RUELLE: *A remark on bound states in potential scattering* (preprint).

it is possible that the probabilities to find particles 2 and 3 in a finite region of their relative-motion configuration space do not tend to zero (*).

Intuitively we should say that such a situation can only occur if the local potential V^{23} does not have a sufficiently regular decrease at infinity (in fact by requiring either uniform decrease ([4]) of the potentials, or smoothness of their Fourier transform ([5]) (in a form implying fast decrease) IKEBE and FADDEEV have shown independently of each other that this continuous singular part is empty). The most striking consequence of this phenomenon results from Donoghue's analysis. From it one can prove that it is always possible to add to some V^{ij} a finite-rank operator with arbitrarily small coupling constant such that $\mathscr{H}_{cs}(H^{ij})$ turns out to be nonempty. This fact expresses the unstability of the three-particle S-matrix against small two-particle perturbations for which the two-particle S-matrices are continuous ([10b]), revealing its great sensitivity to the choice of potentials.

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APPENDIX I

Let $\mathscr{U}(g)$, $g \in G$ be a strongly continuous unitary representation of an Abelian locally compact group G in a Hilbert space $\mathscr{H}$. Then there exists for $\mathscr{U}(g)$ a generalized spectral decomposition

$$\mathscr{U}(g) = \int_{\hat{G}} x(g)\, \mathrm{d}E(x)\,, \tag{I.1}$$

(*) This can happen if f^{23} is a « weak » but not « strong » flight vector ([21]); such vectors can be shown to exist if V^{23} is a rank-one operator constructed along the lines indicated in ex. (1) of DONOGHUE ([22]); for this one must use Stone's theorem ([23]) on the unitary equivalence of self-adjoint operators with simple spectra and same spectral type. In Donoghue's example it is then sufficient to take as singular measure, the ternary Cantor measure.

([21]) K. JACOBS: *Lecture Notes on Ergodic Theory* (Aarhus, 1963).

([22]) W. F. DONOGHUE jr.: *Com. Pure and Appl. Math.*, **18**, 559 (1965).

([23]) M. STONE: *Linear transformations in Hilbert space* (*Amer. Math. Soc. Colloqium Pub*, vol. **15**, New York (1932)).

They are given by

$$W_{\pm}^{\alpha} = s \lim_{t \to \pm\infty} \exp\left[iH_C^{\{N\}}t\right] \exp\left[-iH_{0C}^{\alpha}t\right] P^{\alpha} .$$

Here $H_C^{\{N\}} = H_0^{\{N\}} + \sum_{i<j} (e_i e_j / |x^{ij}| + I^{ij})$, where I^{ij} is square integrable, and $H_{0C}^{\alpha}(t)$ is the one-parameter family of self-adjoint operators defined for $D(\alpha)$ $\{C_1; C_2\}$ by

$$H_{0C}^{\alpha}(t) = H_0^{(C_1,C_2)} + \frac{\varepsilon(t)\, m^{(C_1,C_2)}}{|p^{(C_1,C_2)}|} e_{C_1} e_{C_2} \log\left(2|t| H_0^{(C_1,C_2)}\right) ;$$

here e_C is the total charge of cluster C and $\varepsilon(t) = \pm 1$ for $t \gtrless 0$. Furthermore if $N = 2$ and the interaction is purely Coulomb then W_+ is given by

$$W_{\pm} = \int \mathrm{d}p |\psi_p^{\pm})(p| , \tag{19}$$

where $p)$ denotes a plane wave with momentum p and $|\psi_p^{\pm})$ are the stationary scattering solutions of the Coulomb potential with outgoing or ingoing momentum equal to p. The physical meaning of these « wave operators » is that although the wave packets

$$f_t(p^{(C_1,C_2)}) = (P^{\alpha} \exp\left[-iH_C^{\{N\}}t\right] f)(p^{(C_1,C_2)})$$

do not behave asymptotically like free wave packets, the momentum distributions $|f_t(p^{(C_1,C_2)})|^2$ tend to the asymptotic limits $|(W_{\pm}^{*} f)(p^{(C_1,C_2)})|^2$ which are in fact the only things needed for the computation of cross-sections. However for this last purpose one needs a practicable from of $W_{\pm}^{\alpha}$; for this we split the interaction V_{sc}^{α} into the sum

$$V_{sc}^{\alpha} = V_{sc}^{\alpha s} + \frac{e_{C_1} e_{C_2}}{|x^{(C_1,C_2)}|} ,$$

where the « short-range » part $V_{sc}^{\alpha s}$ comes from the contribution to V_{sc}^{α} of the I^{ij} and of the « multipole » moments of the charge distributions $|\varphi^{b(C)}(x^C)|^2$, which are finite by theorem 2 since the interactions is purely local, and give contributions in $O(1/|x^{(C_1,C_2)}|^2)$ to V_{sc}^{α}. We now use the observation that Kato's proof of the transitivity theorem K.2 remains valid if one of the wave operators in theorem K.2, say $\Omega_+(H_0'; H_0)$, is replaced by a distorted wave operator intertwining the absolutely continuous parts of H_0 and H_0'. Using that extension of theorem K.2 we can prove the existence of

$$\Omega_+\left(H_0^{(C_1,C_2)} + V_{sc}^{\alpha};\ H_0^{(C_1,C_2)} + \frac{e_{C_1} e_{C_2}}{|x^{(C_1,C_2)}|}\right) ,$$

One has

$$f = f_a + f_{cs} + f_d \tag{I.4}$$

and it is easily seen that $\mu^{f_a} = \mu^f_a$, $\mu^{f_{cs}} = \mu^f_{cs}$ and $\mu^{f_d} = \mu^f_d$. The unicity of the decomposition (I.4) is a consequence of the fact that $\mathscr{M}^{(\mathscr{U})}_a$, $\mathscr{M}^{(\mathscr{U})}_{cs}$, $\mathscr{M}^{(\mathscr{U})}_d$ are orthogonal to each other. Let for example f be in $\mathscr{M}^{(\mathscr{U})}_d$, h be in $\mathscr{M}^{(\mathscr{U})}_{cs}$. There exists a denumerable set Z such that $\mu^h(CZ) = 0$ and since μ^f is continuous we have $\mu^f(Z) = 0$. Hence $(h, f)_{\mathscr{H}} = (f, E(Z)h)_{\mathscr{H}} = (E(Z)f, h)_{\mathscr{H}} = 0$. The same kind of proof for $\mathscr{M}^{(\mathscr{U})}_a$ leads with (I.4) to the decomposition (I.2). $\mathscr{M}^{(\mathscr{U})}_0$ being the set of f such that μ^f is continuous, one has the following decompositions:

$$\mathscr{H} = \mathscr{M}^{(\mathscr{U})}_c \oplus \mathscr{M}^{(\mathscr{U})}_d , \tag{1.5}$$

$$\mathscr{H} = \mathscr{M}^{(\mathscr{U})}_a \oplus \mathscr{M}^{(\mathscr{U})}_s . \tag{1.6}$$

Now let $\mathscr{U}_1$ and $\mathscr{U}_2$ be two unitary representations of G in $\mathscr{H}_1$ and $\mathscr{H}_2$, $E_1(\Delta)$ and $E_2(\Delta)$ being the corresponding spectral measures. The tensor product $\mathscr{U}_1 \otimes \mathscr{U}_2$ in $\mathscr{H}_1 \otimes \mathscr{H}_2$ also is a continuous representation and hence admits a spectral decomposition. We denote by $E_{1*2}(\Delta)$ its spectral measure. A simple calculation, using (I.1) and Fourier transforms then shows that for the total set of elements $f_1 \otimes f_2$ in $\mathscr{H}_1 \otimes \mathscr{H}_2$ one has the following identity:

$$\mu^{f_1 \otimes f_2}_{1*2} = \mu^{f_1}_1 * \mu^{f_2}_2 , \tag{I.7}$$

where the star denotes the convolution of measures. As a consequence we have the

Theorem A.I.

i) $\mathscr{M}^{(\mathscr{U}_1 \otimes \mathscr{U}_2)}_d = \mathscr{M}^{(\mathscr{U}_1)}_d \otimes \mathscr{M}^{(\mathscr{U}_2)}_d$,

ii) $\mathscr{M}^{(\mathscr{U}_1 \otimes \mathscr{U}_2)}_c = [\mathscr{M}^{(\mathscr{U}_1)}_c \otimes \mathscr{M}^{(\mathscr{U}_2)}_c] \oplus [\mathscr{M}^{(\mathscr{U}_1)}_c \otimes \mathscr{M}^{(\mathscr{U}_2)}_d] \oplus [\mathscr{M}^{(\mathscr{U}_1)}_d \otimes \mathscr{M}^{(\mathscr{U}_2)}_c]$,

iii) $\mathscr{M}^{(\mathscr{U}_1 \otimes \mathscr{U}_2)}_a \subset [\mathscr{M}^{(\mathscr{U}_1)}_a \otimes \mathscr{M}^{(\mathscr{U}_2)}_a] \oplus [\mathscr{M}^{(\mathscr{U}_1)}_a \otimes \mathscr{M}^{(\mathscr{U}_2)}_s] \oplus [\mathscr{M}^{(\mathscr{U}_1)}_s \otimes \mathscr{M}^{(\mathscr{U}_2)}_a]$.

In particular if $\mathscr{M}^{(\mathscr{U}_1)}_a = \mathscr{H}_1$ or $\mathscr{M}^{(\mathscr{U}_2)}_a = \mathscr{H}_2$ then

$$\mathscr{M}^{(\mathscr{U}_1 \otimes \mathscr{U}_2)}_a = \mathscr{H}_1 \otimes \mathscr{H}_2 .$$

Proof.

$$\mathscr{H}_1 \otimes \mathscr{H}_2 = [\mathscr{M}^{(\mathscr{U}_1)}_c \oplus \mathscr{M}^{(\mathscr{U}_1)}_d] \otimes [\mathscr{M}^{(\mathscr{U}_2)}_c \oplus \mathscr{M}^{(\mathscr{U}_2)}_d] = \tag{I.8}$$

$$= [\mathscr{M}^{(\mathscr{U}_1)}_c \otimes \mathscr{M}^{(\mathscr{U}_2)}_c] \oplus [\mathscr{M}^{(\mathscr{U}_1)}_c \otimes \mathscr{M}^{(\mathscr{U}_2)}_d] \oplus [\mathscr{M}^{(\mathscr{U}_1)}_d \otimes \mathscr{M}^{(\mathscr{U}_2)}_c] \oplus [\mathscr{M}^{(\mathscr{U}_1)}_d \otimes \mathscr{M}^{(\mathscr{U}_2)}_d] ,$$

$M_d(\hat{G})$ being a sub*algebra and $M_r(\hat{G})$ being a closed two-sided ideal in $M(\hat{G})$ one has the inclusions in i) and ii) as a consequence of (I.7) and the fact that $\mathscr{M}^{(\mathscr{U}_1 \otimes \mathscr{U}_2)}_c$ and $\mathscr{M}^{(\mathscr{U}_1 \otimes \mathscr{U}_2)}_d$ are closed subspaces of $\mathscr{H}_1 \otimes \mathscr{H}_2$. These inclusions are in fact equalities by (I.5) and (I.8).

In the same way iii) results from (I.7) and the fact that $M_a(\hat{G})$ is a closed two-sided ideal in $M(\hat{G})$. But we do not have equality, for $M_{cs}(\hat{G})$ is not a sub*algebra of $M(\hat{G})$. Hence $[\mathscr{M}_{cs}^{(\mathscr{U}_1)} \otimes \mathscr{M}_{cs}^{(\mathscr{U}_2)}] \cap [\mathscr{M}_a^{(\mathscr{U}_1 \otimes \mathscr{U}_2)}]$ can be nonempty. The last assertion is a consequence of (I.6). If $\mathscr{U}$ is a continuous one-parameter unitary group, generated by a self-adjoint operator A, the above subspaces will also be denoted by $\mathscr{M}_a(A)$, $\mathscr{M}_{cs}(A)$, $\mathscr{M}_d(A)$, respectively.

Appendix II

We prove here a result due to Ekstein expressing the fact that there is no scattering in the limit of infinitely large displacement of the scatterer and the incident particle. It has been shown rigorously by Cook for square-integrable potentials ([25]); we give a proof for the class of potentials satisfying existence theorem K.1.

Lemma A.II. Denote by $\mathscr{U}(r)$, $r \in R^3$, the unitary representation of the translation group in $L^2(R^3)$. Let $\Omega_\pm = \Omega_\pm(H_0 + V; H_0)$ where $V \in \mathscr{T}(H_0; L^2(R^3))$; then $s \lim_{|x| \to \infty} (1 - \Omega_\pm)\mathscr{U}(x) = 0$.

Proof. It is sufficient to prove the lemma on the set of translates of the Fourier transform of the function $(z - p^2/2m)^{-1} \exp[-p^2/2]$, $\operatorname{Im} z \neq 0$; this function vanishes nowhere so by Wiener's theorem this set is total in $L^2(R^3)$. Let G_0 be the centered Gaussian; then these translates are

$$g_{x_0}(x) = \big(\mathscr{U}(x_0) R_z(H_0) G_0\big)(x) .$$

It results from the partial isometry of $\Omega_\pm$ that

$$\text{(II.1)} \qquad \|(\Omega_\pm - 1)\mathscr{U}(x) g_{x_0}\|^2 =$$

$$= \pm 2i \operatorname{Re} \int_0^{\pm\infty} \big(A \exp[-iH_0 t]\mathscr{U}(x + x_0) G_0,\ B\Omega_- \exp[-iH_0 t]\mathscr{U}(x + x_0) G_0\big)\, \mathrm{d}t .$$

Here we have used a factorization $B^* A$ of $R_z(H_0 + V) V R_z(H_0)$ in which A and B are chosen in the Hilbert-Schmidt class. Let $A = \sum_n a_n |f_n\rangle\langle h_n|$ be the spectral decomposition of A and $\big(\sum_n |a_n|^2\big)^{\frac{1}{2}}$ be its Hilbert-Schmidt norm.

([25]) J. M. Cook: *Proof of a lemma due to Ekstein* (unpublished).

Then

$$\text{(II.2)} \qquad \int_0^{+\infty} \| A \exp[-iH_0 t] \mathscr{U}(x - x_0) G_0 \|^2 \, \mathrm{d}t \leq \int_{-\infty}^{+\infty} \sum_n |a_n|^2 \left| (h_n, \exp[-iH_0 t] \mathscr{U}(x - x_0) G_0) \right|^2 \mathrm{d}t .$$

By Parseval's equality one has

$$\text{(II.3)} \qquad \int_{-\infty}^{+\infty} |(h_n, \exp[-iH_0 t] \mathscr{U}(x + x_0) G_0)|^2 \, \mathrm{d}t = 2\pi \int_{-\infty}^{+\infty} \left| \frac{\mathrm{d}}{\mathrm{d}\lambda} (E(\lambda) h_n, \mathscr{U}(x + x_0) G_0) \right|^2 \mathrm{d}\lambda ,$$

where $E(\lambda)$ is the spectral family of H_0. Now, as is proved in [10b] the following inequality holds:

$$\text{(II.4)} \qquad \frac{\mathrm{d}}{\mathrm{d}\lambda} (E(\lambda) h_n, \mathscr{U}(x - x_0) G_0)^2 \leq \frac{\mathrm{d}}{\mathrm{d}\lambda} \| E(\lambda) h_n \|^2 \frac{\mathrm{d}}{\mathrm{d}\lambda} \| E(\lambda) G_0 \|^2 \leqslant C \frac{\mathrm{d}}{\mathrm{d}\lambda} \| E(\lambda) h_n \|^2 .$$

The integral (II.3) is then majorized by $C\|h_n\|^2$; the convergence to zero of (II.2) as $|x|$ tends to infinity will then result from that of (II.3) which can be derived from Lebesgue's theorem if we can prove that

$$s \lim_{|x| \to \infty} \left| \frac{\mathrm{d}}{\mathrm{d}\lambda} (E(\lambda) h_n, \mathscr{U}(x) G_0) \right|^2 = 0 \qquad \text{a.e.}$$

But a simple calculation gives

$$\left| \frac{\mathrm{d}}{\mathrm{d}\lambda} (E(\lambda) h_n, \mathscr{U}(x) G_0) \right|^2 \leqslant C^2 \left| \int \exp[i\sqrt{2m\lambda}|x| \cos\theta] \, h_n(\sqrt{\lambda}, \theta, \varphi) \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi \right|^2 .$$

The convergence to zero for almost all λ of the right integral is then a consequence of Riemann-Lebesgue lemma. In order to get the convergence to zero of (II.1) it is now sufficient to repeat these arguments for $B\Omega_-$ instead of A and use twice the Schwartz's inequalities.

Appendix III

The next lemma asserts that after a bound state has moved freely during an infinite time, the particles with which it is constitued cannot scatter any more a particle localized in a finite region.

Lemma A.III. Suppose that the interactions V^{ij} are in $\mathscr{T}(H_0^{ij}; \mathscr{H}^{ij})$; then for any bound state $b(jk)$ and $(ijk) = (123)$

$$s \lim_{t \to \pm\infty} (\Omega_\pm^{ij*} - 1) \exp[-iH_0^{b(jk)} t_P] P^{b(jk)} = 0 . \tag{III.1}$$

Proof. This lemma will be a consequence of the more general property

$$s \lim_{t \to \pm\infty} (\Omega_\pm^{ij} - 1) \exp[-iH_0^{(i)} t] = 0 ,$$

which we shall prove on the total set of translates

$$g_z = \mathscr{U}(x) R_z(H_0^{ij}) G_0 , \qquad \operatorname{Im} z \neq 0 .$$

Here $G_0 = G_0^{ij} \otimes G_0^{(k)}$ is the centred Gaussian and $\mathscr{U}(x)$, $x \in R^6$, is the representation of the translation group in $\mathscr{H}^{ij} \otimes \mathscr{H}^{(k)}$ that is

$$\mathscr{U}(x) = \mathscr{U}^{ij}(x^{ij}) \otimes \mathscr{U}^{(k)}(x^{(k)}) , \qquad x = (x^{ij}, x^{(k)}) . \tag{III.2}$$

Let us factorize $R_z(H^{ij}) V^{ij} R_z(H_0^{ij})$ as $B^{ij*} A^{ij}$ where A^{ij} and B^{ij} are Hilbert-Schmidt operators on $\mathscr{H}^{ij}$. Then we get as in lemma A.II

$$\begin{aligned} \| (\Omega_\pm^{ij} - 1) \exp[-iH_0^{(i)} t] R_z(H_0^{ij}) G_0 \|^2 \leqslant \\ \leqslant 2 \int_0^{\pm\infty} \| A^{ij} \exp[-iH_0^{ij} t'] \exp[-iH_0^{(i)} t] G_0 \|^2 \mathrm{d}t' \cdot \\ \cdot \int_0^{\pm\infty} \| (B^{ij} \Omega_\pm^{ij}) \exp[-iH_0^{ij} t''] \exp[-iH_0^{i} t] G_0 \|^2 \mathrm{d}t'' . \end{aligned} \tag{III.3}$$

Now $H_0^{(i)}$ can be decomposed as follows:

$$H_0^{(i)} = \frac{m^{(i,j)}}{m^{(i)}} H_0^{ij} + \frac{m^{(k)}}{m^{(i)}} n^{(i,j)} H_0^{(k)} - 2 \frac{n^{(i,j)}}{m^{(i)}} p^{ij} p^{(k)} ,$$

where $n^{(i,j)} = m_i/(m_i + m_j)$. Using then the fact that t' and t in (III.3) are simultaneously positive or negative we get for the right factors of this inequality

estimates like

$$\text{(III.4)} \qquad \int_0^\infty \| A^{ij} \exp[-iH_0^{ij} t'] \exp[-iH_0^{(i)} t] G_0 \|^2 \mathrm{d}t \leqslant$$

$$\leqslant \int_0^{\pm\infty} \mathrm{d}\tau \int \mathrm{d}p^{(k)} \exp[-|p^{(k)}|^2] \left\| A^{ij} \exp[-iH_0^{ij} t] \mathscr{U}^{ij} \left(\frac{2n^{(i,j)}}{m^{(i)}} p^{(k)} t \right) G_0^{ij} \right\|^2 .$$

Using the same methods as in lemma A.II, in particular inequalities (II.2) and (II.4) we can majorize the integrand of (III.4) by an integrable function; since furthermore this integrand tends to zero with t (see again lemma A.II), Lebesgue's theorem can be applied with the result that the left member of (III.3) tends to zero as t tends to infinity. In order to get a similar conclusion for any g_x it is sufficient to replace in the above proof A^{ij} by $A^{ij} \mathscr{U}^{ij}(x^{ij})$.

Appendix IV

Explicit form of V_{sc}^{α} for local two-particle interactions.

For $D(\alpha) = \{C_1; C_2\}$ one has $V^\alpha = \sum_{\substack{i \in C_1 \\ j \in C_2}} \Gamma^{ij}$. We consider one term Γ^{12} in this sum; we have then

$$\text{(IV.1)} \qquad x^{12} = n^{(1,\sigma_1'')} x^{(1,\sigma_1'')} + n^{(2,\sigma_2'')} x^{(2,\sigma_2'')} + x^{(\sigma_1,\sigma_2)} ,$$

where $C_i'' = C_i - i$ and $n^{(i,\sigma_i'')} = m_i / m_{\sigma_i''}$. A simple calculation gives on the other hand the following contribution of Γ^{12} to V_{sc}^{α}:

$$\text{(IV.2)} \qquad \Gamma_{sc}^{12}(x^{(\sigma_1,\sigma_2)}) = \int |\varphi^{b(\sigma_1)}(x^{\sigma_1})|^2 |\varphi^{b(\sigma_2)}(x^{\sigma_2})|^2 \Gamma^{12}(x^{12}) \, \mathrm{d}x^{\sigma_1} \mathrm{d}x^{\sigma_2} ,$$

which can be written, using $\mathrm{d}x^{\sigma_i} = \mathrm{d}x^{\sigma_i''} \mathrm{d}x^{(i,\sigma_i'')}$, as

$$\text{(IV.3)} \qquad \Gamma_{sc}^{12}(x^{(\sigma_1,\sigma_2)}) = \int d^{b(\sigma_1)}(x^{(1,\sigma_1'')}) \, d^{b(\sigma_2)}(x^{(2,\sigma_2'')}) \, \Gamma^{12}(x^{12}) \, \mathrm{d}x^{(1,\sigma_1'')} \mathrm{d}x^{(2,\sigma_2'')} ,$$

where $d^{b(\sigma_i)}(x^{(i,\sigma_i'')})$ is the probability (or mass) distribution of particle i in the

bound state $b(C_i)$

$$d^{b(C_i)}(x^{(i,C_i'')}) = \int |\varphi^{b(C_i)}(x^{C_i})|^2 \mathrm{d}x^{C_i''} . \tag{IV.4}$$

Formula (IV.3) is the basis of our interpretation of V_{sc}^{α} as the interaction between two inhomogeneous mass distributions.

Appendix V

A nonunitary multichannel S-matrix.

We consider a three-particle system with a Hamiltonian $H = H_0 + V^{123} + V^{23}$ where V^{23} is such that $\Omega_{\pm}^{23}$ exist and are complete but $\mathscr{M}_{cs}(H^{23})$ is nonempty (*); and V^{123} is a three-particle interaction, such that $\Omega_{\pm}(H; H_0 + V^{23})$ exist and are complete. Our aim is to show that for a certain choice of V^{123}, the subspace $\mathscr{M}_{cs}(H^{23})$ does not reduce the operator $S(H; H_0 + V^{23}) = \Omega_{+}^{*}(H; H_0 + V^{23}) \cdot \Omega_{-}(H; H_0 + V^{23})$. Our construction of V^{123} copies that of ([26]). Let us define the orthogonal subspaces of $\mathscr{H}$ (reducing $H_0 + V^{23}$):

$$\mathscr{H}_1 = \mathscr{H}^{(1)} \otimes \mathscr{M}_{cs}(H^{23}) , \qquad \mathscr{H}_2 = \mathscr{H}^{(1)} \otimes \mathscr{M}_{a}(H^{23}) .$$

According to theorem A.I one has $\mathscr{H}_1 \subset \mathscr{M}_a(H_0 + V^{23})$; on the other hand, owing to the completeness of $\Omega_{\pm}^{23}$ and lemma 6 the restriction of $H_0 + V^{23}$ to $\mathscr{H}_2$ is unitarily equivalent to H_0. Then it is not difficult to see that for any $h_1 \in \mathscr{H}_1$, there exists $h_2 \in \mathscr{H}_2$ such that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \| E^{(23)}(\lambda) h_1 \|^2 = \frac{\mathrm{d}}{\mathrm{d}\lambda} \| E^{(23)}(\lambda) h_2 \|^2 = \frac{\mathrm{d}}{\mathrm{d}\lambda} \nu_0(\lambda) ,$$

where $E^{(23)}(\lambda)$ is the spectral family of $H_0 + V^{23}$. Denote by $\mathscr{M}(h_1)$ and $\mathscr{M}(h_2)$ the minimal subspaces reducing $H_0 + V^{23}$ and containing respectively h_1 and h_2. The restrictions of $H_0 + V^{23}$ to these subspaces have a simple spectrum and the same spectral type; according to Stone ([23]) they are unitarily equivalent. On the other hand any vector f_i in one of these subspaces is obtained by applying to h_i some function f of $H_0 + V^{23}$, where f is square integrable with respect to $\mathrm{d}\nu_0(\lambda)$. Let us consider in

$$\mathscr{M} = \mathscr{M}(h_1) \oplus \mathscr{M}(h_2)$$

(*) Such V^{23} are provided for example through Donoghue's analysis.

([26]) T. Kato and S. T. Kuroda: *Nuovo Cimento*, **14**, 1102 (1959).

the following subspaces:

$$\mathscr{M}_+ = \{f(H_0 + V^{23})h_1 \oplus f(H_0 + V^{23})h_2;\ f \in L^2(R, \mathrm{d}\nu_0)\}\ ,$$
$$\mathscr{M}_- = \{f(H_0 + V^{23})h_1 \oplus -f(H_0 + V^{23})h_2;\ f \in L^2(R, \mathrm{d}\nu_0)\}\ .$$

These subspaces are orthogonal to each other and reduce $H_0 + V^{23}$ which has on them a restriction with a simple spectrum. Let us choose some $f_+ \in \mathscr{M}_+$; then

$$V^{123} = |f_+)(f_+|$$

has the properties required at the beginning. To see this we remark that $S(H; H_0 + V^{23})$ is simply the identity on $\mathscr{M}_-$, while on $\mathscr{M}_+$ it is some function $\exp[is(H_0 + V^{23})]$ of $H_0 + V^{23}$. This function is real and different from one in general ([26]). Now let $f_1 = f(H_0 + V^{23})h_1$, $f_1 \in \mathscr{H}_1$; a simple calculation then shows

$$S(H; H_0 + V^{23})f_1 = \frac{\exp[is(H_0 + V^{23})] + 1}{2} f(H_0 + V^{23})h_1 + \\ + \frac{\exp[is(H_0 + V^{23})] - 1}{2} f(H_0 + V^{23})h_2$$

and this concludes the proof since $S(H; H_0 + V^{23})f_1$ has a nonzero component in H_2.

RIASSUNTO (*)

Si studiano i problemi matematici che si riscontrano nella teoria non relativistica dello scattering a molti canali con l'aiuto del lavoro di Kato sulla teoria astratta dello scattering dipendente dal tempo. Si considerano soprattutto le reazioni a due corpi nei sistemi di molte particelle; si ricava rigorosamente una forma fisica intuitiva per i corrispondenti operatori d'onda e si dimostra una completezza asintotica parziale sotto a soglia di tre corpi. Si studia anche la rilevanza della rappresentazione dipendente dal tempo per uno studio completo e dettagliato dei sistemi a tre particelle. Si stabiliscono alcuni risultati ausiliari sugli stati legati a molte particelle.

(*) *Traduzione a cura della Redazione.*

Приближение, зависящее от времени, для нерелятивистского многоканального рассеяния.

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